Complex Numbers in Geometry

Evan O'Dorney

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1. Let $ABCD$ be any quadrilateral, and let W, X, Y, Z be the centers of squares erected outwardly on the sides. Prove that $WY = XZ$.

2. The convex quadrilateral $ABCD$ is inscribed in a circle centered at O and its diagonals intersect at E. Prove that if the midpoints of AD, BC, and OE are collinear, then either $AB = CD$ or $\angle AEB = 90$.

3. Let $ABCD$ be a convex quadrilateral inscribed in a circle. Let M and N be the midpoints of AB and CD, and let E and F be the intersections of AD and BC and of AC and BD. Prove that

$$\frac{2MN}{EF} = \left|\frac{AB}{CD} - \frac{CD}{AB}\right|.$$

4. In the same diagram, prove that the circumcircle of $\triangle FMN$ is tangent to EF.

5. Trapezoid $ABCD$, with $\overline{AB}||\overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.