Berkeley Math Circle — October 5, 2010

Algebra!

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Formal Outline

1 Sets

- Natural Numbers
- Integers
- Real Numbers
- and Beyond

2 Numbers

- Fractions
- Decimals
- Powers
- Radicals
- Different Bases

3 Things with VARIABLES

- Formulas
- Functions
- Sequences
- Polynomials

4 Equations

- Linear
- Quadratic
- And Beyond

Informal Outline

1 Problems vs. Exercises

2 Thinking logically

3 Thinking hard

4 Having fun

Exercises AND problems

1 *Important Sequences*. Master the following sequences (i.e., learn by heart the first dozen or so terms, and compute lots more, and get so acquainted with it that it becomes a friend):

(a) Squares

(b) Cubes

(c) Triangular numbers

(d) Powers of two

(e) Powers of three

(f) Perfect Powers

(g) The repunits

(h) Factorials

(i) Fibonacci numbers

(j) Pascal's Triangle

2 *Two Classic Word Problems*.

(a) If a chicken-and-a-half lay an egg-and-a-half in a day-and-a-half, how many eggs does one chicken lay in one day? Formulate and solve the general problem!

(b) Pat walks to school going 4 miles per hour. Pat returns along the same route, but goes 6 miles per hour. What is Pat's average speed? Formulate and solve the general problem!

3 *A Darwinian Struggle*. At time $t = 0$ minutes, a virus is placed into a colony of 2,010 bacteria. Every minute, each virus destroys one bacterium apiece, after which all the bacteria and viruses divide in two. For example, at $t = 1$, there will be $2009 \times 2 = 4018$ bacteria and 2 viruses. At $t = 2$, there will be 4016×2 bacteria and 4 viruses, etc. Will the bacteria be driven to extinction? If so, when will this happen?

4 Compute the following. No writing and no calculator allowed!

(a) 35^2

(b) 53^2

5 *Sums*.

(a) Find the sum of the first 100 natural numbers.

(b) Find the sum of the divisors of 100.

6 Define $f(x) = 1/(1-x)$. What is the value of

$$f(f(f(f(f(\cdots(2010)))\cdots),$$

where the expression above has 2010 pairs of parentheses?

7 For each integer $n > 1$, find *distinct* positive integers x and y such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

8 *Conjecture, test, prove!* Notice the following:

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 5^2 + 12^2 &= 13^2 \\ 7^2 + 24^2 &= 25^2 \\ 9^2 + 40^2 &= 41^2 \end{aligned}$$

Conjecture a pattern. Test it. Prove it!

9 *Triangular Numbers*

(a) Are triangular numbers ever perfect squares?

(b) Let T be a triangular number. What can you say about $8T + 1$? Why?

(c) Let T be a triangular number. What can you say about $9T + 1$? Why?

(d) Investigate and come up with your own conjecture. Can you prove it?

10 *Conjecture, test, prove!*

$$\begin{aligned} 6^2 - 5^2 &= 11 \\ 56^2 - 45^2 &= 1111 \\ 556^2 - 445^2 &= 111111 \end{aligned}$$

11 Let $\mathbf{N}$ denote the natural numbers $\{1, 2, 3, 4, \ldots\}$. Consider a function f that satisfies $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$ for all $n \in \mathbf{N}$. Find a nice simple algorithm for $f(n)$. Your algorithm should be a single sentence long, at most.

12 What is the value of the *infinite continued fraction*

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

13 *Conjecture, test, prove!*

$$\left\lfloor\sqrt{44}\right\rfloor = 6,\ \left\lfloor\sqrt{4444}\right\rfloor = 66, \dots$$

14 *Conjecture, test, prove!*

$$49, 4489, 444889, 44448889, \dots$$

15 For each positive integer n, find positive integer solutions $x_1, x_2, \dots, x_n$ to the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1.$$

16 The function $f : \mathbf{Z} \to \mathbf{Z}$ satisfies $f(n) = n - 3$ if $n \geq 1000$ and $f(n) = f(f(n+5))$ if $n < 1000$. Find $f(84)$.

17 If $x^2 + y^2 + z^2 = 49$ and $x + y + z = x^3 + y^3 + z^3 = 7$, find xyz.

18 Find all real values of x that satisfy $(16x^2 - 9)^3 + (9x^2 - 16)^3 = (25x^2 - 25)^3$.

19 Find all ordered pairs of positive integers (x, y) that satisfy $x^3 - y^3 = 721$.

20 Determine the triples of integers (x, y, z) satisfying the equation

$$x^3 + y^3 + z^3 = (x + y + z)^3.$$

21 Compute

$$\frac{(10^4 + 324)(22^4 + 324) \cdots (58^4 + 324)}{(4^4 + 324)(16^4 + 324) \cdots (52^4 + 324)}.$$

22 Solve!

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$
$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$
$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$