Berkeley Math Circle 1/11/11

Algebra? (aka Algebra IV)

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1 A monk climbs a mountain. He starts at 8AM and reaches the summit at noon. He spends the night on the summit. The next morning, he leaves the summit at 8AM and descends by the same route that he used the day before, reaching the bottom at noon. Must there be an instant in time between 8AM and noon at which the monk was at exactly the same spot on the mountain on both days? (Notice that we do not specify anything about the speed that the monk travels. For example, he could race at 1000 miles per hour for the first few minutes, then sit still for hours, then travel backward, etc. Nor does the monk have to travel at the same speeds going up as going down.)

2 Pat works in the city and lives in the suburbs with Sal. Every afternoon, Pat gets on a train that arrives at the suburban station at exactly 5PM. Sal leaves the house before 5 and drives at a constant speed so as to arrive at the train station at exactly 5PM to pick up Pat. The route that Sal drives never changes.

One day, this routine is interrupted, because there is a power failure at work. Pat gets to leave early, and catches a train which arrives at the suburban station at 4PM. Instead of phoning Sal to ask for an earlier pickup, Pat decides to get a little exercise, and begins walking home along the route that Sal drives, knowing that eventually Sal will intercept Pat, and then will make a U-turn, and they will head home together in the car. This is indeed what happens, and Pat ends up arriving at home 10 minutes earlier than on a normal day. Assuming that Pat's walking speed is constant, that the U-turn takes no time, and that Sal's driving speed is constant, for how many minutes did Pat walk?

3 A person dives from a bridge into a river and swims upstream through the water for 1 hour at constant speed. She then turns around and swims downstream through the water at the same rate of speed. As the swimmer passes under the bridge, a bystander tells her that her hat fell into the river as she originally dived. The swimmer continues downstream at the same rate of speed, catching up with the hat at another bridge exactly 1 mile downstream from the first one. What is the speed of the current in miles per hour?

4 Two towns, A and B, are connected by a road. At sunrise, Pat begins biking from A to B along this road, while simultaneously Dana begins biking from B to A. Each person bikes at a constant speed, and they cross paths at noon. Pat reaches B at 5PM while Dana reaches A at 11:15PM. When was sunrise?

5 Bay Area Rapid Food sells chicken nuggets. You can buy packages of seven or packages of 11. What is the largest integer n such that there is no way to buy exactly n nuggets? Can you generalize this?

6 A bug is crawling on the coordinate plane from $(7,11)$ to $(-17,-3)$. The bug travels at constant speed one unit per second everywhere but quadrant II (negative x- and positive y-coordinates), where it travels at $\frac{1}{2}$ unit per second. What path should the bug take to complete its journey in minimal time? Generalize!

7 What is the first time after 12 o'clock at which the hour and minute hands meet?

8 Sonia walks up an escalator which is going up. When she walks at one step per second, it takes her 20 steps to get to the top. If she walks at two steps per second, it takes her 32 steps to get to the top. She never skips over any steps. How many steps does the escalator have?

9 Find (and then, if possible, prove) a formula for the sum of the first n perfect cubes.

10 A column of soldiers one mile long begins marching at a constant speed. A general starts walking when the soldiers do, inspecting the troops. The general starts at the rear of the column (the last soldier) and begins marching, at constant speed, toward the front, eventually reaching the first soldier at the front. Then the general instantly turns around, and walks back toward the rear of the column. When the general reaches the rear of the column, the last soldier is now at the spot occupied by the first soldier when the column began marching. If the soldiers march at one mile per hour, how fast does the general march?

11 Let a and b be integers greater than one which have no common divisors. Prove that

$$\sum_{i=1}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor = \sum_{j=1}^{a-1} \left\lfloor \frac{bj}{a} \right\rfloor,$$

and find the value of this common sum.

12 How many distinct terms are there when

$$(1+x^7+x^{13})^{100}$$

is multiplied out and simplified?

13 Several marbles are placed on a circular track of circumference one meter. The width of the track and the radii of the marbles are negligible. Each marble is randomly given an orientation, clockwise or counterclockwise. At time zero, each marble begins to travel with speed one meter per minute, where the direction of travel depends on the orientation. Whenever two marbles collide, they bounce back with no change in speed, obeying the laws of inelastic collision.

What can you say about the possible locations of the marbles after one minute, with respect to their original positions? There are three factors to consider: the number of marbles, their initial locations, and their initial orientations.