Sequences, or
1, 2, 3, 4, 5, 6, 7, 8, what comes next?

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Problem 1. Let's start with 1, 2, 3, 4, 5, 6, 7, 8. I hope we all agree that 9 comes next. Why?

Problem 2. Think of at least two sequences that go 1, 2, 3, 4, 5, 6, 7, 8, 9, and then stop.

Problem 3. Think of at least two sequences that go 1, 2, 3, 4, 5, 6, 7, 8, 9, 8, …

Problem 4. Think of at least two sequences that go 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, …

The real point of these is not the cute math involved in the particular sequences I have in mind, but rather that the next number could be **anything**. Sequence puzzles are not math problems; they are human problems, getting into the mind of the person who wrote the puzzle, or knowing where the pattern came from. A "good" solution is only better than a "bad" one if we have similar taste in mathematical simplicity.

It's always better to think about a sequence that arises from a real mathematical problem. It could certainly be based on a problem from geometry or from combinatorics, for example!

Problem 5. Into how many pieces can you divide a circular pizza with 1 straight cut? With 2? With 3? With 4? We'll assume your cuts must go all the way across, that you cannot rearrange the pieces in between cuts, and that you want to make the most pieces possible.

Problem 6. Into how many pieces can you divide a circular pizza by labeling 2 points on the edge and then connecting them? With 3 points, connected in all possible ways? 4 points? 5 points? 6? Does it depend on how you arrange the points?

There may be some surprises in these sequences! Still, if you don't know where a sequence comes from, there's a general rule for making the next number in a sequence puzzle anything you want.

The idea: write the differences of the numbers you have in your sequence, to make a "difference sequence" and then write the differences of those to make the "second differences" and so on, until you get to a point where you have a constant (or you have just one number, in which case you can guess it might be constant perhaps). This method is called **finite differences**. The goal is whether we can take the long description that produces just one term at a time and convert it to a formula that lets us figure out any term we want more quickly.

Problem 7. Given the sequence $a_n = 1, 4, 7, 10, 13, \ldots$, I hope you can all see that the differences $Da_n = 3, 3, 3, 3, \ldots$. That helps you calculate the next term, so do that, and then also use it to find a general formula for a_n in terms of n.

Problem 8. If the first term of a sequence is 7, and it has first difference a constant 2, write a formula for the sequence.

Problem 9. How about a formula for the sequence 2, 9, 22, 41, 66, 97, 134, …?

Problem 10. If the first term of a sequence is 11, the first term of its first differences is 5, and the second difference is a constant 12, write a formula for the sequence.

Problem 11. What about 0, 6, 24, 60, 120, 210, 336, …? This one might be amenable to a good guess-and-check, too, as well as the finite differences method.

Problem 12. Now let's work the other way: if $a_n = mx + b$, what will Da_n be equal to? What if a_n has a quadratic pattern, maybe $Ax^2 + Bx + C$? Cubic?

Problem 13. If the first term of a sequence is w, the first term of its first differences is x, the first term of its second differences is y, and the third difference is a constant z, write a formula for the sequence.

Problem 14. Instead of powers of x like x^2 and x^3, the **falling** powers are very useful here. We define them as $x^{\underline{2}} = x\,(x-1)$ and $x^{\underline{3}} = x\,(x-1)\,(x-2)$ and so on. Explain why these are convenient.

Problem 15. Now relate what you've learned so far to Pascal's Triangle.

Problem 16. What does all the above stuff about differences tell you about the sum of a sequence? Can you find a formula for the sum of the numbers 1 through n? How about for the sum of the squares? Cubes? Fourth powers?

Problem 17. Make a nice formula for 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, 1941, 2517, 3214, 4048, …

Problem 18. What happens when you try this same technique on 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, …?

Problem 19. What about if you try it on 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …?