Poncelet's Theorem

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§0. Statement of the Theorem

First, some definitions about polygons:

Definition 1: An *n-gon* P is a sequence of n distinct points $(p_0, \ldots, p_{n-1})$ in the plane, called the *vertices* of P. For convenience, set $p_n = p_0$. The line segments $\overline{p_i p_{i+1}}$ for $i = 0, 1, \ldots, n{-}1$ are called the *sides* of P.

Remark: To describe an n-gon P, what we care about is the cyclic order of the points p_i. Thus,

$$P = (p_0, p_1, p_2, p_3) \simeq (p_1, p_2, p_3, p_0) \simeq (p_0, p_3, p_2, p_1),$$

but $(p_0, p_1, p_2, p_3) \not\simeq (p_0, p_2, p_1, p_3)$. (They don't have the same set of sides.)

Definition 2: P is *inscribed in* a curve C if its vertices p_i all lie on C.

Definition 3: P is *circumscribed about* a curve C if its sides $\overline{p_i p_{i+1}}$ are tangent to C.

Poncelet's Theorem: Suppose that E_0 is an ellipse in the plane and E_1 is another ellipse that contains E_0 in its interior. If there is one n-gon P that is both inscribed in E_1 and circumscribed about E_0, then there is an infinite number of such n-gons. (In fact, any point on E_1 is a vertex of exactly one such n-gon.)

Simple case: Let's look at an easy case first: Two concentric circles:

$$E_0: \qquad x^2 + y^2 = 1 \qquad \text{and} \qquad E_1: \qquad x^2 + y^2 = r^2, \quad r > 1.$$

1. What value of r will make it possible to inscribe a 3-gon (i.e., a triangle) in E_1 in such a way that it will be circumscribed about E_0? What can you say about these triangles?

2. What value of r will make it possible to inscribe a 4-gon (i.e., a quadrilateral) in E_1 in such a way that it will be circumscribed about E_0? What can you say about these quadrilaterals?

3. What value of r will make it possible to inscribe a 5-gon in E_1 in such a way that it will be circumscribed about E_0? Is there only one value of r that will work?

4. Can you describe the value(s) of r that you'd need to have an n-gon inscribed in E_1 and circumscribed about E_0? Do the various values have any relation with each other? (Hint: Complex numbers can be useful here, especially DeMoivre's Formula.)

Slightly more complicated case: Suppose E_0 is the unit circle, and E_1 is given as follows

$$E_0: \qquad x^2 + y^2 = 1 \qquad \text{and} \qquad E_1: \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad r > 1.$$

where $a, b > 1$.

5. *Assuming Poncelet's Theorem*, what relationship between a and b will allow a 3-gon to be inscribed in E_1 and circumscribed about E_0? (Check that your answer agrees with the result of the first exercise when $a = b = r > 1$.)

6. *Assuming Poncelet's Theorem*, what relationship between a and b will allow a 4-gon to be inscribed in E_1 and circumscribed about E_0? (Check that your answer agrees with the result of the second exercise when $a = b = r > 1$.)

7. (Harder) *Assuming Poncelet's Theorem*, what relationship between a and b will allow a 5-gon to be inscribed in E_1 and circumscribed about E_0?

§1. On quadratic curves, especially ellipses

Consider a polynomial of degree 2 in the variables x and y:

$$Q(x, y) = A\,x^2 + 2B\,xy + C\,y^2 + 2D\,x + 2E\,y + F.$$

To avoid degenerate cases, assume that not all of A, B, and C are zero. The curve X in the plane defined by the equation $Q(x, y) = 0$ is said to be a curve of the *second degree*.

If λ is a nonzero number, then $Q(x, y) = 0$ if and only if $\lambda Q(x, y) = 0$, so we can replace Q by λQ without changing the curve X.

It's not always easy to see what X looks like, but can help to *normalize* the curve by translating, rotating, and scaling.

Translation. If we set $(x, y) = (\bar{x} + p, \bar{y} + q)$ for some point (p, q), we can write

$$0 = Q(x, y) = Q(\bar{x} + p, \bar{y} + q) = \bar{Q}(\bar{x}, \bar{y})$$

where

$$\bar{Q}(\bar{x}, \bar{y}) = A\,\bar{x}^2 + 2B\,\bar{x}\bar{y} + C\,\bar{y}^2 + 2\bar{D}\,\bar{x} + 2\bar{E}\,\bar{y} + \bar{F}.$$

for some new constants $\bar{D}$, $\bar{E}$, and $\bar{F}$. This is called 'translating to new coordinates'.

We say that (p, q) is the *center* of X if $\bar{D} = \bar{E} = 0$. (You should think of this as 'completing the square', but in two variables instead of one.)

Example: Let $Q = x^2 + xy + y^2 - 4x - 5y$. Then

$$\begin{aligned} x^2 + xy + y^2 - 4x - 5y &= (\bar{x}{+}1)^2 + (\bar{x}{+}1)(\bar{y}{+}2) + (\bar{y}{+}2)^2 - 4(\bar{x}{+}1) - 5(\bar{y}{+}2) \\ &= \bar{x}^2 + \bar{x}\bar{y} + \bar{y}^2 - 7, \end{aligned}$$

so the (a?) center of this curve $Q(x, y) = 0$ is $(p, q) = (1, 2)$.

8. Show that if $AC - B^2 \neq 0$, then there always is a center and it is unique.

ROTATION. One can also rotate coordinates by an angle θ

$$\begin{aligned} x &= \quad \cos\theta\,\bar{x} + \sin\theta\,\bar{y}, \\ y &= -\sin\theta\,\bar{x} + \cos\theta\,\bar{y}, \end{aligned}$$

9. Show that there is always an angle θ so that the rotated polynomial

$$Q(x, y) = Q(\ \cos\theta\,\bar{x} + \sin\theta\,\bar{y}\ ,\ -\sin\theta\,\bar{x} + \cos\theta\,\bar{y}\ ) = \bar{Q}(\bar{x}, \bar{y})$$

has $\bar{B} = 0$. Check that $\bar{A} + \bar{C} = A + C$ and that $\bar{A}\bar{C} - \bar{B}^2 = AC - B^2$.

REMARK: Translation and rotation don't change what a quadratic curve X 'looks like', just where it is positioned in the plane.

When $AC - B^2 \neq 0$, by translating and rotating, we can get down to curves described by equations of the form

$$Q(x, y) = A\,x^2 + C\,y^2 + F = 0,$$

where A and C are non-zero. When A and C have the same sign and F has the opposite sign, we can divide by $-F$ and get down to the case

$$Q(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

where a and b are positive numbers. Of course, when $a = b$ this is a circle of radius a. Otherwise, this is an *ellipse*.

SCALING. We can even get down to a circle if we are willing to *scale* x and y independently:

$$x = a\,\bar{x} \qquad \text{and} \qquad y = b\,\bar{y},$$

so that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \bar{x}^2 + \bar{y}^2 - 1 = 0.$$

AFFINE COORDINATE CHANGES: Translation, Rotation, and Scaling are special cases of the so-called *affine coordinate changes*, the most general of which is

$$\begin{aligned} x &= a\,\bar{x} + b\,\bar{y} + p, \\ y &= c\,\bar{x} + d\,\bar{y} + q, \end{aligned} \qquad \text{where } ad - bc \neq 0.$$

10. Why do you think one needs the condition $ad - bc \neq 0$?

11. Explain why affine changes take linear polynomials $L(x, y) = D\,x + E\,y + F$ and quadratic polynomials $Q(x, y) = A\,x^2 + 2B\,xy + C\,y^2 + 2D\,x + 2E\,y + F$ into linear polynomials and quadratic polynomials, respectively. (In particular, affine changes take lines to lines and ellipses to ellipses. Moreover, affine changes are closed under composition and inverse.)

We are going to need one more basic fact about affine changes of coordinates:

Area change rule: The area of a figure (for example, a triangle) measured in xy-coordinates is $|ac - bd|$ times its area measured in $\bar{x}\bar{y}$-coordinates.

REMARK: The proof of the area change rule relies on the fact that the area of the parallelogram based at $(0, 0)$ generated by the two vectors (a, b) and (c, d) is $|ad - bc|$.

Also, note that, $ad - bc > 0$ if $(0, 0)$, (a, b), and (c, d) is a counterclockwise-enumerated triangle, while $ad - bc < 0$ if $(0, 0)$, (a, b), and (c, d) is a clockwise-enumerated triangle.

TANGENT LINES: A line meets an ellipse in at most 2 points.

12. Prove this. (Hint: You can do this by brute force, but, using affine changes of coordinates, it's enough to show it in the case when the ellipse is the circle $x^2 + y^2 - 1 = 0$ and the line is $x - r = 0$ for some $r \geq 0$. Why?)

A line that meets an ellipse in exactly one point is said to be *tangent* to the ellipse.

Example: The line $\cos\theta\, x + \sin\theta\, y - 1 = 0$ is tangent to the circle $x^2 + y^2 - 1 = 0$ at the point $(\cos\theta, \sin\theta)$.

13. If (x_0, y_0) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that the line

$$\frac{x_0}{a^2}\, x + \frac{y_0}{b^2}\, y - 1 = 0$$

is tangent to the ellipse at the point (x_0, y_0). (Hint: there's more than one way to do this. One way is to parametrize the line in the form $(x, y) = (x_0 + a^2 y_0 t, y_0 - b^2 x_0 t)$ and check that $t = 0$ is the only time this point lies on the ellipse. There is a slicker way, though.)

14. Show that if a point p lies *outside* an ellipse E, then there are exactly two lines tangent to E that pass through p. (Hint: Again, you can do this by brute force, but, using affine changes of coordinates, it's enough to show this in the case that E is the circle $x^2 + y^2 - 1 = 0$ and $p = (r, 0)$ where $r > 1$. Why? In this special case, where are the two points of tangency?)

15. (Important!) Show that if E is an ellipse with center C, X is a point outside E, and P_1 and P_2 are the two points on E whose tangent lines to E pass through X, then the two triangles (C, P_1, X) and (C, P_2, X) have the same area. (Hint: First, consider the case when E is a circle!) (Note, though, that these two triangles are oppositely oriented!)

§2. Projective transformations

This section is harder than the earlier sections, but we need the Normalization Theorem at the end to reduce Poncelet's Theorem to the simpler case of a pair of ellipses with a common center.

It turns out that affine changes of coordinates are not the only transformations that take lines to lines and ellipses to ellipses.

Projective coordinate changes. Consider a set of 9 constants $a_1, \ldots, a_9$ and write

$$x = \frac{a_1\,\bar{x} + a_2\,\bar{y} + a_3}{a_7\,\bar{x} + a_8\,\bar{y} + a_9}, \qquad \text{where} \quad \det\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \neq 0.$$
$$y = \frac{a_4\,\bar{x} + a_5\,\bar{y} + a_6}{a_7\,\bar{x} + a_8\,\bar{y} + a_9},$$

The condition that the determinant not vanish is what you need to be sure that you can solve the above formulae for $\bar{x}$ and $\bar{y}$. In fact, if

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \quad \text{is the inverse matrix of} \quad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

then

$$\bar{x} = \frac{b_1\,x + b_2\,y + b_3}{b_7\,x + b_8\,y + b_9},$$
$$\bar{y} = \frac{b_4\,x + b_5\,y + b_6}{b_7\,x + b_8\,y + b_9},$$

16. Explain why a projective coordinate change as above takes lines to lines, except for the line $b_7\,x + b_8\,y + b_9 = 0$ (which doesn't seem to have anywhere to go). (Hint: Show that

$$D\,x + E\,y + F = \frac{\bar{D}\,\bar{x} + \bar{E}\,\bar{y} + \bar{F}}{a_7\,\bar{x} + a_8\,\bar{y} + a_9}$$

for some constants $\bar{D}$, $\bar{E}$, $\bar{F}$. Why does this do it? What goes wrong when $D\,x + E\,y + F = b_7\,x + b_8\,y + b_9$?

17. Explain why a projective coordinate change as above takes an ellipse E to an ellipse $\bar{E}$, as long as E doesn't meet the line $b_7\,x + b_8\,y + b_9 = 0$. (Question: What happens if E meets this line in one point, or two points?) Explain also why such a coordinate change takes lines tangent to such an E to lines tangent to $\bar{E}$.

The reason we need projective transformations is the following result, which is proved using techniques from Linear Algebra (usually a second-year college course).

Projective center alignment: If E_0 and E_1 are ellipses, with E_0 inside E_1, then there is a projective change that takes E_0 (respectively, E_1) to an ellipse $\bar{E}_0$ (respectively, $\bar{E}_1$) such that $\bar{E}_0$ and $\bar{E}_1$ have the same center C.

Once we know this, proving Poncelet's Theorem reduces to checking the cases

$$E_0 : \quad x^2 + y^2 - 1 = 0, \qquad\qquad E_1 : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad (a, b > 1).$$

§3. Angle measure and other measures

The thing that made Poncelet's Theorem so easy to prove for concentric circles is that all the line segments with endpoints on the outer circle that are tangent to the inner circle have the same length and subtend the same (radian) measure of arc. All you have to do is determine whether this angle is a rational multiple of 2π, and you'll know whether the inscribing/circumscribing polygonal path you draw starting at any point will close. It clearly doesn't depend on the point where you start, which is exactly Poncelet's Theorem in this case.

18. The unit circle $x^2 + y^2 - 1 = 0$ is parametrized by

$$(x, y) = (\cos\theta,\ \sin\theta).$$

Implicit differentiation yields $x\,dx + y\,dy = 0$, and this can be written in the form

$$\frac{dy}{x} = -\frac{dx}{y}.$$

While neither of these expressions is defined everywhere on the circle (because x and y each vanish at two points), show that each ratio (where defined) is $d\theta$, which is defined everywhere on the circle, and integrating it between two points on the circle gives the total angle between the two points.

Now, in the general case, the angle subtended by the line segments that are inscribed in E_1 and are tangent to E_0 is not independent of which segment you choose, so you can't use angle measure to prove Poncelet's Theorem. However (and this is the amazing thing), it turns out that there *is* a sort of 'generalized' angle measure that is the same for all such line segments. This angle measure is a little hard to see, but we are going to describe it and show that it works.

It is not so surprising that other measures of angle are useful. A famous example is one that comes from Kepler's Laws: Remember that Kepler's First Law says that a planet moves on an ellipse, with the Sun at its focus. Kepler's Second Law says that a line segment joining the planet to the Sun sweeps out equal areas in equal times. What this means is that, if you wanted to mark out equal 'months' on a planetary orbit, you wouldn't mark them out with equal angles, nor would you mark them out with equal distances along the orbit. Instead, the measure you would use would be to divide so that the 'elliptical pie wedges' have equal area. In other words, you'd measure according to 'elliptical pie wedge area'.

19. In polar coordinates (r, θ), an ellipse with a focus at the origin takes the form

$$r = \frac{a(1-\epsilon^2)}{(1 - \epsilon\, \cos\theta)},$$

where ϵ is the *eccentricity*. The area swept out between two values of θ is

$$A = \int_{\theta_0}^{\theta_1} \tfrac{1}{2} r(\theta)^2\, d\theta = \frac{a^2(1-\epsilon^2)^2}{2} \int_{\theta_0}^{\theta_1} \frac{d\theta}{(1 - \epsilon\, \cos\theta)^2}.$$

Re-express this in rectangular coordinates and explain what the 'elliptical pie wedge area' represents geometrically.

What we are going to see is that, by distorting the natural angular measure on an ellipse by the right geometric quantity, we can find the right 'generalized angle' measure that will make Poncelet's Theorem easy.

§4. The proof

Here is a different way to think about Poncelet's Theorem: Let E_0 and E_1 be ellipses, with E_0 contained in the interior of E_1. For any point q_0 on E_0, let p_0 be the point on E_1 where the counterclockwise tangent ray to E_0 at q_0 meets the outer ellipse E_1. Since p_0 is exterior to E_0, it lies on two tangents to E_1. One of those is the tangent at q_0 and the other is tangent at another point q_1 on E_0. Now continue this by induction: For each q_i ($i \geq 0$), let p_i be the point on E_1 where the counterclockwise tangent ray at q_i to E_0 meets E_1 and let q_{i+1} be the unique point on E_0 so that the two tangents to E_0 that pass through p_i are tangent to E_0 at q_i and q_{i+1}.

Poncelet's Theorem: If there is some point q_0 on E_0 and an integer $n > 1$ so that $q_n = q_0$, then for *any* point q'_0 on E_0, one has $q'_n = q'_0$.

In other words, the polygonal path inscribed in E_1 and circumscribed about E_0 either closes in n steps for all starting points or does not close in n steps for any starting point.

1. The first step in the proof is to use projective geometry to reduce to the case where the two ellipses are presented in standard form as

$$E_0 = \{ (x, y) \mid x^2 + y^2 = 1 \}$$
$$E_1 = \{ (z, w) \mid z^2/a^2 + w^2/b^2 = 1 \}$$

for some constants $a, b > 1$. This is proved by showing that the corresponding quadratic forms (in 3 variables) can be simultaneously diagonalized. (Of course, this depends on the fact that the two ellipses are disjoint, with one contained in the other.)

2. The second step is to consider the set S consisting of points $(x, y, z, w) \in \mathbf{R}^4$ where (x, y) lies on E_0 while (z, w) lies both on E_1 and on the tangent line to E_0 at (x, y). In other words, the set S is defined by the equations

$$x^2 + y^2 - 1 = \frac{z^2}{a^2} + \frac{w^2}{b^2} - 1 = xz + yw - 1 = 0.$$

Now, S is the disjoint union of two circles:

$$S_+ = \{ (x, y, z, w) \in E \mid xw - yz > 0 \}$$
$$S_- = \{ (x, y, z, w) \in E \mid xw - yz < 0 \}$$

Note that if $q_0 = (x, y)$, then $p_0 = (z, w)$ where (x, y, z, w) lies in S_+. Moreover, $q_1 = (u, v)$ where (u, v, z, w) lies in S_-. In fact, the projection of S onto either E_0 or E_1 is a (trivial) double cover of that ellipse.

Observe that there is a unique map $\tau : S_\pm \to S_\mp$ with the property that $\tau(x, y, z, w) = (x, y, \bar{z}, \bar{w})$ and that there is also a unique map $\sigma : S_\pm \to S_\mp$ that has the property that $\sigma(x, y, z, w) = (\bar{x}, \bar{y}, z, w)$. In other words, τ is the deck transformation for $(x, y) : S \to E_0$ while σ is the deck transformation for $(z, w) : S \to E_1$.

The formulae for τ and σ can be found as follows: Let (x, y, z, w) lie in S. The tangent line to E_0 at (x, y) passes through (z, w) and is parallel to the vector $(y, -x)$, so it follows that, if $\tau(x, y, z, w) = (x, y, \bar{z}, \bar{w})$, then there must be a value t so that

$$\bar{z} = z + ty, \qquad \bar{w} = w - tx.$$

Substituting these relations into the equation $(\bar{z})^2/a^2 + (\bar{w})^2/b^2 = 1$ yields a quadratic equation for t that has $t = 0$ as a root, so the other root must be a rational expression in the coefficients. In fact, computation gives

$$t = -2\,\frac{(zy/a^2 - wx/b^2)}{(y^2/a^2 + x^2/b^2)}\,.$$

Thus, τ is expressed rationally in terms of the functions x, y, z, and w on S.

Similarly, if $\sigma(x, y, z, w) = (\bar{x}, \bar{y}, z, w)$, the line through (x, y) and $(\bar{x}, \bar{y})$ must be orthogonal to the vector (z, w), so there must be a value s so that

$$\bar{x} = x + sw, \qquad \bar{y} = y - sz.$$

Substituting these relations into the equation $(\bar{x})^2 + (\bar{y})^2 = 1$ yields a quadratic equation for s that has $s = 0$ as a root, so the other root must be a rational expression in the coefficients. Computation gives

$$s = -2\,\frac{(xw - yz)}{(z^2 + w^2)}.$$

Thus, σ is expressed rationally in terms of the functions x, y, z, and w on S.

To prepare for the next step, observe that the function $xw - yz$ is *odd* with respect to σ, i.e., that

$$\bar{x}w - \bar{y}z = -(xw - yz).$$

This says that the area of the oriented triangle with vertices $(0,0)$, (x,y), and (z,w) is the negative of the area of the oriented triangle with vertices $(0,0)$, $(\bar{x},\bar{y})$, and (z,w), a fact that is geometrically obvious.

Correspondingly, for τ, one finds

$$\bar{z}y/a^2 - \bar{w}x/b^2 = -(zy/a^2 - wx/b^2).$$

This can be checked by hand, but it also has a natural interpretation in terms of the *dual ellipse* E_1^* defined by the equation $a^2\,z^2 + b^2\,w^2 - 1 = 0$.

20. Figure out this geometric interpretation by considering the two tangent lines to E_1^* that pass through (x,y). (Hint: You can find them easily since you know the points (z,w) and $(\bar{z},\bar{w})$ on E_1.)

Something like this might have been expected, since $(xw - yz)$ is the expression whose sign distinguishes the components S_+ and S_-. One might also note that the function $zy/a^2 - wx/b^2$ does not vanish on S (do you see why not?) and that it has opposite signs on the two components.

3. The third step involves investigating the differentials on S. Differentiating the defining equations of S yields three differential relations, which can be written together as

$$\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & z/a^2 & w/b^2 \\ z & w & x & y \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dw \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to check that the coefficient matrix has rank 3 at every point of S, which implies, by the Implicit Function Theorem, that S is a smoothly embedded curve in $\mathbf{R}^4$. Moreover, it implies that there are relations of the form

$$\frac{dx}{y(zy/a^2 - wx/b^2)} = \frac{-dy}{x(zy/a^2 - wx/b^2)} = \frac{-b^2\,dz}{w(xw - yz)} = \frac{a^2\,dw}{z(xw - yz)}.$$

Now, you may object that these differential expressions are not well-defined everywhere since, after all, x, y, z, and w each vanish somewhere on S. However, notice that they do not all *simultaneously* vanish and hence, at every point of S, at least one of these differential expressions is well-defined and smooth. In other words, there is a well-defined 1-form $d\theta$ on S that equals each of these four expressions on the open set where that expression is well-defined. Moreover, by the implicit function theorem, this 1-form $d\theta$ is nowhere vanishing.

Now, because $\tau(x,y,z,w) = (x,y,\bar{z},\bar{w})$, and because the expression $(zy/a^2 - wx/b^2)$ is *odd* with respect to τ, it follows from the first two expressions for $d\theta$ that $d\theta$ is odd with respect to τ, i.e., that $\tau^*(d\theta) = -d\theta$. Similarly, because $\sigma(x,y,z,w) = (\bar{x},\bar{y},z,w)$ and because $(xw - yz)$ is odd with respect to σ, it follows that $d\theta$ is odd with respect to σ, i.e., $\sigma^*(d\theta) = -d\theta$.

In particular, it follows that $d\theta$ is invariant under $\tau \circ \sigma$ and $\sigma \circ \tau$ (which is the inverse of $\tau \circ \sigma$). In other words, if we choose to parametrize E_+ (say) with respect to $d\theta$, then $\sigma \circ \tau$ is just rotation by a fixed amount α.

More precisely, orient S so that $d\theta$ is a positive 1-form and define

$$L = \int_{S_+} d\theta\,.$$

Then we can define a mapping $\theta : S_+ \to \mathbf{R}/(L \cdot \mathbf{Z})$ by setting

$$\theta(x,y,z,w) = \left(\int_{(1,0,z_0,w_0)}^{(x,y,z,w)} d\theta \right) \qquad \mathrm{mod}\ L.$$

where $(1,0,z_0,w_0)$ lies on E_+. This identifies S_+ with the circle in such a way that $\sigma \circ \tau$ is carried over into translation by some α in $\mathbf{R}/(L \cdot \mathbf{Z})$.

Now, the condition of the polygon closing in n steps is exactly the condition that $n\alpha \equiv 0$ in $\mathbf{R}/(L \cdot \mathbf{Z})$. Note that this doesn't depend on the starting point q_0, so Poncelet's Theorem is proved.