

# Plane Geometry I:

## *Reflection Across a Line Does It!*

Berkeley Math Circle – Beginners

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*Note:* We shall work on the problems below over several circle sessions. Bring this handout with you to all Geometry I sessions of BMC–beginners. Try to understand what the problems say and draw pictures for them as best as you can. You are not expected to be able to solve the problems on your own, at least not in the beginning of the Geometry series at BMC–beginners. Some of the problems in this handout are from “Mathematical Olympiads”, part II by Stoyan Budurov and Dimo Serafimov, State Publishing Company “Narodna Prosveta”, Sofia, 1985.

### 1. OVERARCHING PROBLEMS

- (1) (**Three Squares**) Three identical squares with bases  $AM$ ,  $MH$ , and  $HB$  are put next to each other to form a rectangle  $ABCD$ . Find the sum of the angles  $\angle AMD + \angle AHD + \angle ABD$  and prove that your answer is correct.
- (2) (**Farmer and Cow**) During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible? Prove your claim.

### 2. EXTENSIONS

- (3) (**Shortest Broken Line**) Two lines  $p_1$  and  $p_2$  intersect. Two points  $A$  and  $B$  lie in the acute angle formed by the lines. Find a point  $C$  on  $p_1$  and a point  $D$  on  $p_2$  so that the broken line  $ADBCA$  has the smallest possible length. Prove that the points you have found indeed yield this smallest possible length.
- (4) (**Locating Angle Bisector**) Two points  $A$  and  $B$  and a line  $l$  are given so that the line intersects segment  $AB$  (neither  $A$  nor  $B$  lies on  $l$ ). Find point  $C$  on  $l$  so that the angle bisector of  $\angle ACB$  lies on  $l$ . Prove that your construction is correct.

### 3. PREPARATION WITH REFLECTIONS: HOW TO DRAW AND BASIC PROPERTIES

- (5) **(Parallel axes of reflection)** Line  $g_1$  and  $g_2$  are parallel and point  $M$  does not lie on any of them.

(a) Find the reflection  $M_1$  of  $M$  across  $g_1$  and the reflection  $M_2$  of  $M$  across  $g_2$ .

(b) If  $d$  is the (shortest) distance between  $g_1$  and  $g_2$ , prove that  $|M_1M_2| = 2d$ .

Consider the case when  $M$  is between the two parallel lines and the case when it is not between them.

- (6) **(Reflections on a circle)** Two lines  $g_1$  and  $g_2$  intersect in point  $O$ , and point  $M$  does not lie on any of the lines. Let  $M_1$  be the reflection of  $M$  across  $g_1$ ,  $M_2$  the reflection of  $M_1$  across  $g_2$ , and  $M_3$  the reflection of  $M$  across  $g_2$ . Prove that

(a) points  $M$ ,  $M_1$ ,  $M_2$ , and  $M_3$  lie on a circle with center  $O$ .

(b) the line determined by  $O$  and the midpoint of  $M_2M_3$  is perpendicular to  $M_2M_3$ .

**Theorem 1.** *Segments preserve their length under reflection across a line. In other words, if we reflect segment  $AB$  across line  $l$  to segment  $A_1B_1$ , then  $|AB| = |A_1B_1|$ .*

**Theorem 2.** *In  $\triangle ABC$  the segment connecting  $C$  to the midpoint  $M$  of side  $AB$  is perpendicular to  $AB$  if and only if  $\triangle ABC$  is isosceles with  $|AC| = |BC|$ .*

- (7) **(Reflections and angles)** Two lines  $g_1$  and  $g_2$  intersect in point  $O$ , and point  $M$  does not lie on any of the lines.

(a) Draw the reflection  $M_1$  of  $M$  across  $g_1$  and the reflection  $M_2$  of  $M_1$  across  $g_2$ .

(b) In case  $O$  does not lie on line  $MM_2$ , prove that  $\angle M_2MO = \angle MM_2O$ .

(c) If  $|OM| = 9$  cm and  $\angle MOM_2 = 120^\circ$ , find the distance from  $O$  to line  $MM_2$ .

**Theorem 3.** *Angles are preserved under reflection across a line. In other words, if three points  $A$ ,  $B$  and  $C$  are reflected across line  $l$  to points  $A_1$ ,  $B_1$  and  $C_1$ , then  $\angle ABC = \angle A_1B_1C_1$ .*

**Theorem 4.** *In a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle, the shorter leg is half of the hypotenuse. (Note: The shorter leg lies against the  $30^\circ$  angle. Why?)*

- (8) **(Reflections and rectangle)** Lines  $m$  and  $n$  are mutually perpendicular and intersect in point  $A$ . Point  $B$  does not lie on any of these lines but is in their plane. Point  $B_1$  is the reflection of  $B$  across  $m$ , point  $B_2$  is the reflection of  $B_1$  across  $n$ , and point  $B_3$  is the reflection of  $B_2$  across  $m$ .

(a) Draw points  $B_1$ ,  $B_2$ , and  $B_3$ .

(b) Prove that points  $B$ ,  $B_1$ ,  $B_2$ , and  $B_3$  are the vertices of a rectangle.

(c) If point  $B$  is at a distance 1.5 cm from line  $n$  and segment  $AB$  makes an angle of  $30^\circ$  with line  $n$ , find the length of  $AB$ .

**Theorem 5.** *A quadrilateral is a rectangle if and only if its diagonals intersect each other in a point equidistant from all four vertices. In other words, a quadrilateral  $ABCD$  is a rectangle exactly when its diagonals  $AC$  and  $BD$  intersect in point  $O$  such that  $|OA| = |OB| = |OC| = |OD|$ .*

- (9) **(Reflections and angle bisector)** In  $\triangle ABC$  the angle bisector of  $\angle BAC$  lies on line  $g_1$ , while line  $g_2$  is perpendicular to  $g_1$  and passes through point  $A$ .
- Prove that the angle bisector of the supplementary angle to  $\angle BAC$  lies on  $g_2$ .
  - If  $B_1$  and  $B_2$  are the reflections of  $B$  across line  $g_1$  and  $g_2$ , respectively, prove that  $B_1$  and  $B_2$  lie on line  $AC$ .
  - Find the length of segment  $BB_1$ , given  $|AB| = 2.5$  cm and  $\angle AB_2B = 30^\circ$ .

**Theorem 6.** *Let  $\alpha$  and  $\beta$  be two supplementary angles, i.e.,  $\alpha$  and  $\beta$  share one ray, and their other rays are opposite to each other. Then the angle bisectors of  $\alpha$  and  $\beta$  are perpendicular to each other.*

- (10) **(Reflecting a whole line)** Given are lines  $a$ ,  $b$ , and  $s$ . On line  $b$  draw point  $A_1$  so that its reflection across  $s$  lies on  $a$ . After that locate this reflection of  $A_1$  on line  $a$ .

**Theorem 7.** *The reflection of a line  $b$  across a line  $s$  is another line  $b_1$ . Moreover, either both lines  $b$  and  $b_1$  make the same angle with  $s$ , or they are both parallel to  $s$ .*

#### 4. RELATED EXERCISES FOR WORK AT THE MATH CIRCLE AND AT HOME

- (11) What is *reflection across line  $s$* ? Describe in words.
- (12) How do we draw reflections of points across  $s$ ? Draw several examples. Are there any special cases we need to consider? What instruments do we need to draw such reflections precisely? (*Hint:* The answer here includes, among other things, being able to drop a perpendicular from a point to a line.)
- (13) There are essentially *three* cases in Problem 5: two *outside* cases and one *inside* case. Why does the problem ask you to consider only two cases? Explain.
- (14) Theorem 2 has two directions:
- You assume that  $|AC| = |BC|$  and then prove that  $CM$  is perpendicular to  $AB$ ; the latter is written as  $CM \perp AB$ .
  - You assume that  $CM \perp AB$  and then prove that  $|AC| = |BC|$ .
- Prove each directions. Of course, you are allowed to use and obliged to state which criteria for congruent triangles you use.
- (15) What is a *circle*? Describe carefully and precisely, in words.
- (16) Given segment  $AB$ , what instruments do you need to find precisely the midpoint  $M$  of  $AB$ ? Describe the construction step by step. (*Hint:* The previous exercise on Theorem 2 essentially gives away the idea: you need to construct an extra special point  $C$ .)
- (17) Prove Theorem 1. Consider separately four cases, depending on whether segment  $AB$  and line  $l$  intersect and whether they are perpendicular to each other.

- (18) Prove Theorem 3. For a full and formal solution, you may need to consider a bunch of cases depending on the relative position of  $\angle ABC$  with respect to (wrt) line  $l$ . Just concentrate on a generic case, e.g., all three points are on one side of  $l$  and none of the sides of  $\triangle ABC$  is perpendicular to  $l$ .
- (19) **(For the die-hards)** Prove Theorem 4. (*Hint:* One way is to pick the midpoint of the hypotenuse, show that it is the center of the circle passing through the vertices of the triangle, and see why this introduces a new equilateral triangle in your picture.)
- (20) Prove Theorem 5. Note that it requires the proof of two directions.
- (21) Draw two supplementary angles neither of which is right. Draw their angle bisectors and notice that they seem perpendicular to each other. Prove Theorem 6.
- (22) What is an *angle bisector* of  $\angle ABC$ ? Describe carefully in words. Assuming we are forbidden to use a protractor, what *other* instruments do we need to draw the angle bisector of angle? Describe the construction step by step.
- (23) Can you think of *two different* solutions to Problem 10?
- (24) Investigate what happens in the special positions for the lines  $a$ ,  $b$  and  $s$  in Problem 10. Does the solution to Problem 10 change? Why or why not? Explain.
- (25) Find out and describe all positions of the lines  $a$ ,  $b$  and  $s$  in which Problem 10 has **no** solution. Explain in words what goes wrong and why there is no solution. (*Hint:* One way to look at it is that a certain isosceles triangle will cause trouble. There is another way to describe this troublesome situation, using the word *reflection*.)
- (26) **(For the experienced)** Prove Theorem 7.
- (27) **(For the die-hard explorers)** Suppose in Problem 5 we keep on reflecting:

$$M \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_2 \xrightarrow{g_1} M_3 \xrightarrow{g_2} M_4 \xrightarrow{g_1} \dots$$

What is the distance  $|MM_{2010}|$ ? How about  $|MM_{2011}|$ ? What additional initial data do you need to know to determine these distances? Explain.

- (28) What does it mean that 3 lines are in *general* position?
- (29) What special positions for 3 lines can you think of?
- (30) What happens if we reflect *all* points of a line  $b$  across  $s$ ? What figure do you get? Do we need to reflect *all* points of  $b$  across  $s$  in order to find where  $b$  goes?

## 5. FURTHER APPLICATIONS OF REFLECTIONS

- (31) **(Minimal perimeter, again!)** Given  $\triangle ABC$ , on the ray opposite to ray  $\overrightarrow{CA}$  take a point  $B_1$  so that  $|CB_1| = |CB|$ . Prove that
- (a) Point  $B_1$  is the reflection of  $B$  across the angle bisector  $l$  of the exterior angle of the triangle at vertex  $C$ .
  - (b) If  $D$  is an arbitrary point on  $l$  different from  $C$ , the perimeter of  $\triangle ABD$  is bigger than the perimeter of  $\triangle ABC$ .

- (32) **(Find the perimeter!)** In  $\triangle ABC$ ,  $|AC| = |BC|$  and  $|AB| = 10$  cm. Through the midpoint  $D$  of  $AC$  we draw a line perpendicular to  $AC$ . This line intersects  $BC$  in point  $E$ . The perimeter of  $\triangle ABC$  is 40 cm. Find the perimeter of  $\triangle ABE$ .
- (33) **(Medians and angle bisector)** In isosceles  $\triangle ABC$  ( $|AC| = |BC|$ ) the medians to the congruent sides intersect in point  $D$ . Prove that segment  $DC$  divides the angle of the triangle at  $C$  into two equal parts.
- (34) **(A fifth criteria for triangle congruence)** Prove that if in a given triangle one of its sides, an angle adjacent to that side, and the sum of the other two sides of the triangle are correspondingly congruent to the same elements in a second triangle, then the two triangles are congruent.