(K)not Symmetry ...

1. When something is rigid, such as a square or a triangle, it is easy to find all of its symmetries. This is not the case for knots. Imagine that the following figure was made from a loop of rope. Does it have any symmetry?

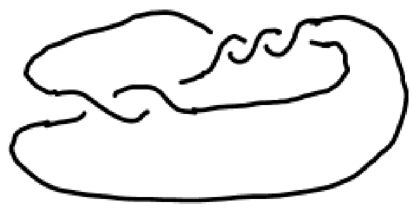

Figure 1: The 5_2 knot.

2. A symmetry is a rigid motion of space that carries points with rope to points with rope. How many symmetries does the loop below on the left have? What about the loop below on the right? (Don't forget to count the symmetry that does not move anything.)

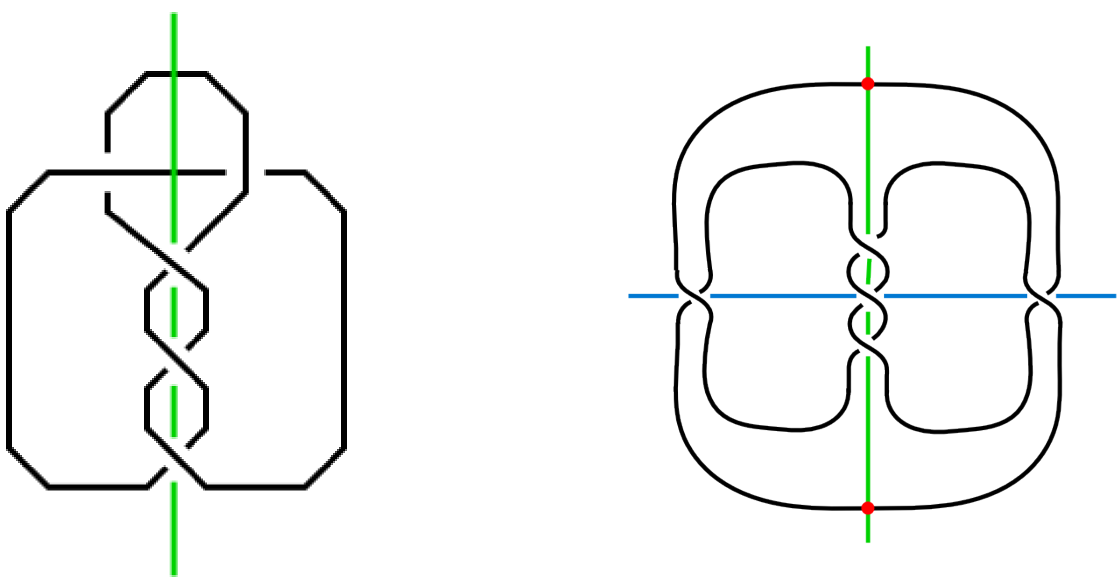

Figure 2: Full symmetry of the 5_2

3. Show how to deform the loop from problem 1 until it matches the the loop on the left and then the loop on the right from problem 2.

4. Play the (k)not symmetry game with a friend and the knots on the attached page (or make up and use your own knots.) To play the game, rearrange the knot to find symmetries. Your score is the number of symmetries that you find. You can keep going with more knots, and set a goal of finding 12 symmetries.

5. What is the most symmetries you can find in one knot? What is a knot with the smallest number of symmetries?

6. Use induction to prove that every rational tangle has four symmetries. Does this help you play the (k)not symmetry game?

7. A *rank* 2 *lattice* is a discrete spanning subgroup of $\mathbb{R}^2$. A *Dirichlet domain* is a set of points that are at least as close to one lattice point as they are to any other lattice point. Draw Dirichlet domains for the following lattices. How many orientation-preserving symmetries does each one have? How many orientation-preserving symmetries fixing a lattice point does each lattice have?

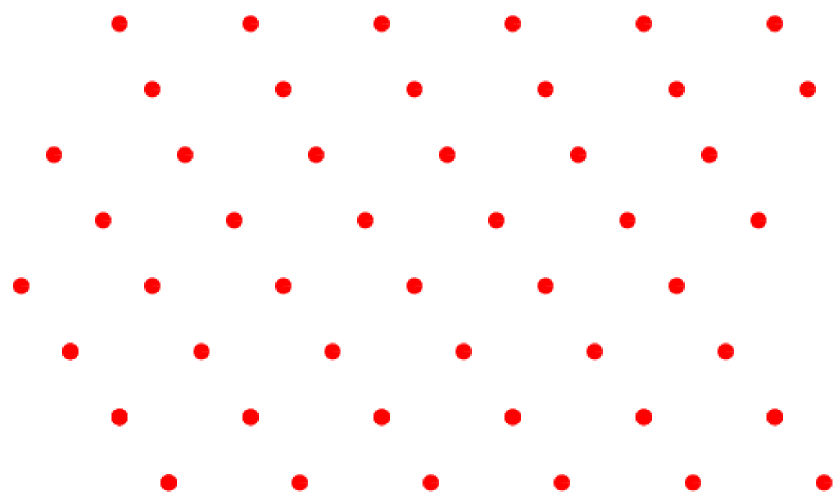

Figure 3: The $\frac{5}{8} + \frac{1}{2}i$ lattice

8. We can label any point in the plane with an ordered pair (a, b). If we want to think of the plane as a "number plane," we can write this as $a + bi$. To multiply one of these "numbers" by $a + bi$, we just scale it by the length of $a + bi$ and rotate it by the angle that $a + bi$ makes with 1 as viewed from zero.

   (a) What is ii?

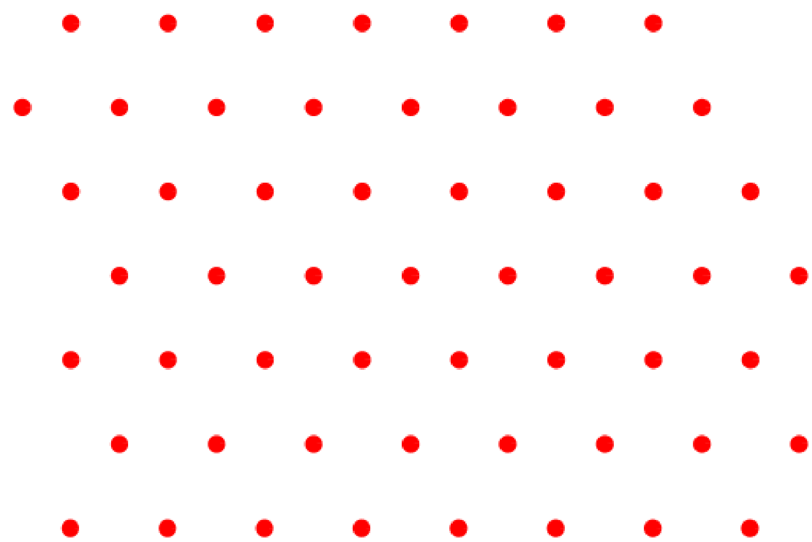

Figure 4: The ω lattice

(b) What is $(2+3i)+(5+6i)$?

(c) What is $(-2+5i)(4+6i)$?

9. If we only consider lattices up to similitude, then we can assume that one vector in a generating set is 1 and the other vector in the generating set has positive i-component. Call the second vector τ. Explain why the lattice labeled by τ is the same as the one labeled by $\tau+1$. Explain why the lattice labeled by τ is the same as the one labeled by $-1/\tau$. (Hint: consider what happens if the vector denoted by τ is scaled and rotated to 1.) The transformation $\tau \mapsto \tau+1$ is denoted by T. The transformation $\tau \mapsto -1/\tau$ is denoted by S. (Compare these with the moves in the Rational Tangle Dance.) The group generated by these two transformations is called the *modular group* and it is denoted by $\mathrm{PSL}_2(\mathbb{Z})$.

10. If $\tau = a+bi$, set $\bar{\tau} := a-bi$. If we define the distance between σ and τ to be

$$d(\sigma,\tau) := \ln\left(\frac{|\sigma-\bar{\tau}|+|\sigma-\tau|}{|\sigma-\bar{\tau}|-|\sigma-\tau|}\right),$$

then the following is a Dirichlet domain for the modular group. Find the center point of this domain.

11. Show that it is possible to get to any rational number by a sequence of S and T transformations starting with zero. (Hint: think about how to send any rational number to zero by a sequence of S and T^{-1} transformations.)

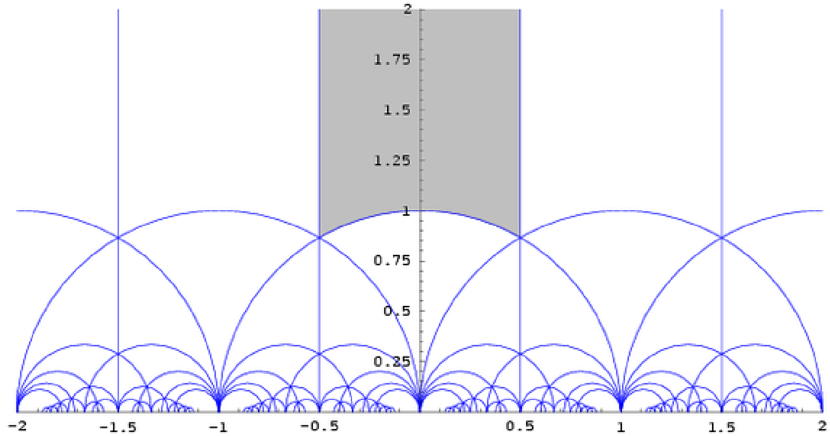

Figure 5: A Dirichlet domain for the modular group

12. What is S^2? What is $(ST)^3$. The figure above can be used to show that any relation is a product of a sequence of these two relations. This shows that the rational tangles really are labeled by the (extended) rational numbers, but this is a different story.

13. Find an ideal polyhedral decomposition of the complement of the 5_2 knot.