FUNCTIONAL EQUATIONS

1. Cauchy's Equation

In early 19th century, Cauchy studied the following equation[1]

$$f(x+y) = f(x) + f(y).$$

It turns out that this equation has a single family of solutions over rational numbers and an extremely weird set of solutions over real numbers. However, amazingly enough all of these weird solutions can be discarded if we make one of many common assumptions about f. We will see all these next.

1.1. **Proof for Rationals.** First put $y = 0$:

$$\begin{aligned} f(x+0) &= f(x) + f(0) \\ f(0) &= 0 \end{aligned}$$

Then put $y = -x$:

$$\begin{aligned} f(x-x) &= f(x) + f(-x) \\ f(-x) &= -f(x) \qquad \forall x \in \mathbb{Q} \end{aligned}$$

Then, by repeated application of the original equation to expand the right side of $f(nx) = f(x + x + \cdots + x)$ we get

$$f(nx) = nf(x) \qquad \forall x \in \mathbb{Q}, \forall n \in \mathbb{N}$$

By substituting $x = \frac{1}{n}$:

$$f\left(\frac{1}{n}\right) = \frac{1}{n} f(1) \qquad \forall n \in \mathbb{N}$$

Combining the two equations above with $x = \frac{1}{m}$. we get :

$$f\left(\frac{n}{m}\right) = nf\left(\frac{1}{m}\right) = n\frac{1}{m}f(1) = \frac{n}{m}f(1) \qquad \forall m, n \in \mathbb{N}$$

Using $f(-x) = -f(x)$ and multiplying the equation above by -1, we get

$$\begin{aligned} -f\left(\frac{n}{m}\right) &= -\frac{n}{m}f(1) \\ f\left(-\frac{n}{m}\right) &= \left(-\frac{n}{m}\right) \cdot f(1) \qquad \forall m, n \in \mathbb{N} \\ f(q) &= qf(1) \qquad \forall q \in \mathbb{Q} \end{aligned}$$

Thus, we have found that $f(x) = cx \quad \forall x \in \mathbb{Q}$ and some constant $c \in \mathbb{R}$. It is obvious that this family of functions is indeed a solution of $f(x+y) = f(x) + f(y)$ for rational x and y. More generally, it is easy to show that

$$f(\alpha q) = qf(\alpha) \qquad \forall q \in \mathbb{Q}, \alpha \in \mathbb{R}$$

1.2. **Weirdness over Reals.** In case you are curious, the more appropriate name for "weirdness" in math is "pathology". Thus, in this section we want to show that all solutions of Cauchy equation, distint from $f(x) = cx$, are extremely pathological over reals. To show this, we will use a notion of a *dense* subset in $\mathbb{R}^2$ - a subset of $\mathbb{R}^2$ is called dense if any disk in $\mathbb{R}^2$ (however small) contains a point of this subset. The subset that we will be considering is the graph of $y = f(x)$.

Without loss of generality, assume that $f(q) = q \quad \forall q \in \mathbb{Q}$ and $f(\alpha) \neq \alpha$ for some $\alpha \in \mathbb{R}$. Let $f(\alpha) = \alpha + \delta$, $\delta \neq 0$. We show how to find a point in an arbitrary circle C centred at (x, y) with radius r, where $x, y \in \mathbb{Q}$ and $r \in \mathbb{R}, r > 0$.

Let $\beta = \frac{y-x}{\delta}$ and choose a rational number $b \neq 0$ close to β such that

$$|\beta - b| < \frac{r}{4|\delta|}$$

[1]Much of the material in this handout have been adopted from wikipedia.org and imomath.com.

Choose a rational number a close to α such that:

$$|\alpha - a| < \frac{r}{2\,|b|}$$

Now, let

$$\begin{aligned} X &= x + b(\alpha - a) \\ Y &= f(X) \end{aligned}$$

We now show that the point (X, Y), which is part of the graph $y = f(x)$ lies inside C.

$$\begin{aligned} Y &= f(x + b(\alpha - a)) \\ &= f(x) + f(b(\alpha - a)) \\ &= f(x) + bf(\alpha - a) \\ &= f(x) + b(f(\alpha) - f(a)) \\ &= f(x) + bf(\alpha) - bf(a) \\ &= x + bf(\alpha) - ba \\ &= y - \delta\beta + bf(\alpha) - ba \\ &= y - \delta\beta + b(\alpha + \delta) - ba \\ &= y + b(\alpha - a) - \delta(\beta - b) \end{aligned}$$

Using the expressions for X and Y, we now show that each of them is close to the center of C.

$$\begin{aligned} X &= x + b(\alpha - a) \le x + |b||\alpha - a| < x + r/4 \\ X &= x + b(\alpha - a) \ge x - |b||\alpha - a| > x - r/4 \end{aligned}$$

Thus, $|X - x| < r/4$.

$$\begin{aligned} Y &= y + b(\alpha - a) - \delta(\beta - b) \le y + |b(\alpha - a)| + |\delta(\beta - b)| < y + r/4 + r/4 = y + r/2 \\ Y &= y + b(\alpha - a) - \delta(\beta - b) \ge y - |b(\alpha - a)| - |\delta(\beta - b)| > y - r/4 - r/4 = y - r/2 \end{aligned}$$

Thus, $|Y - y| < r/2$. It is now obvious that (X, Y) lies in C. We just need to show that any circle with $x, y \in \mathbb{R}$ (not necessarily in $\mathbb{Q}$) has a point of the graph. Take any such circle C_1 centered at (x_1, y_1) with radius r_1. We can find a pair of points $x', y' \in \mathbb{Q}$ such that the distance between (x_1, y_1) and (x', y') is less than $r_1/2$. Furthermore, we can pick a radius $r' < r_1/2$ and construct a circle C' centered at (x', y') with radius r'. This circle lies inside C_1 and we know that the graph has a point in it. Thus, C_1 also contains a point of the graph.

1.3. **Additional Assumtions.** There is a number of assumptions that, if assumed, force f to be $f(x) = cx$. The first one is the assumption that f is *bounded* on some interval. If it is bounded, we can take a circle in this interval beyond the bounds that won't contain any point of the $y = f(x)$ graph.

The second assumption is that f is continuous at a point. If f is continuous at a point, it is bounded around that point. Thus, the first assumption is a corollary of the second.

The third assumption is taht f is monotonic on an interval. If f is decreasing (increasing), we can take a circle in this interval and above(below) the graph. This circle won't contain any point of the $y = f(x)$ graph.

1.4. **Equations Reducible to Cauchy.** There are a few equations that can be easily reduced to Cauchy's equation. These include:

- $f : \mathbb{R} \to (0, +\infty),\ f(x + y) = f(x)f(y)$. Function $g(x) = \log f(x)$ satisfies Cauchy's equation.
- $f : (0, +\infty) \to \mathbb{R},\ f(xy) = f(x) + f(y)$. Function $g(x) = f(a^x)$ satisfies Cauchy's equation.
- $f : (0, +\infty) \to (0, +\infty),\ f(xy) = f(x)f(y)$. Function $g(x) = \log f(a^x)$ satisfies Cauchy's equation.

2. Example Problem

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for which $f(xy) = f(x)f(y) - f(x + y) + 1$. Solve the same problem for the case $f : R \to R$.