AIME PREPARATION HANDOUT

NGOC MAI TRAN

These are selected problems from past AIMEs (American Invitational Mathematics Examination), divided into three categories: warm-ups, intermediate and advanced. Do attempt problems from at least two categories. The goal is to enjoy yourself and learn something new. Have fun!

1. Warm-ups

Question 1 of 2007 AIME:. How many positive perfect squares less than 10^6 are multiples of 24?

Question 3 of 2000 AIME:. In the expansion of $(ax+b)^{2000}$, where a, b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a+b$.

Question 7 of 2007 AIME:. Let

$$N = \sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor).$$

Find the remainder when N is divided by 1000. Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denotes the least integer that is greater than or equal to x.

2. Intermediate

Question 8 of 2000 AIME2: In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB = \sqrt{11}$ and $AD = \sqrt{1001}$, find BC^2.

Question 9 of 2001 AIME2: Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is $\frac{m}{n}$, where m and n relatively prime positive integers. Find $m+n$.

3. Advanced

Question 13 of 2003 AIME2: A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Department of Statistics, UC Berkeley, CA 94703, USA
E-mail address: tran@stat.berkeley.edu.
Date: March 8^{th}, 2011.

Question 14 of 2007 AIME:. Let a sequence be defined as follows: $a_1 = 3, a_2 = 3$, and for $n \geq 2$, $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\dfrac{a_{2007}^2 + a_{2006}^2}{a_{2007}a_{2006}}$.

Question 14 of 2000 AIME2: Every positive integer k has a unique factorial base expansion $(f_1, f_2, \ldots, f_m)$, meaning that

$$k = 1!f_1 + 2!f_2 + 3!f_3 + \ldots + m!f_m$$

where each f_i is an integer, $0 \leq f_i \leq i$, and $0 < f_m$. Given that $(f_1, f_2, \ldots, f_j)$ is the factorial base expansion of

$$16! - 32! + 48! - 64! + \ldots + 1968! - 1984! + 2000!,$$

find the value of $f_1 - f_2 + f_3 - f_4 + \ldots + (-1)^{j+1}f_j$.