Berkeley Math Circle
Monthly Contest 8 – Solutions

1. Let x and y be positive integers. Can $x^2 + 2y$ and $y^2 + 2x$ both be squares?

   Solution. The answer is no. Because of the symmetry between x and y, we can assume that $x \geq y$. Then

   $$x^2 < x^2 + 2y \leq x^2 + 2x < x^2 + 2x + 1 = (x+1)^2.$$

   Thus $x^2 + 2y$, which lies between two consecutive squares, cannot be a square.

2. On a 5×5 chessboard, a king moves according to the following rules:

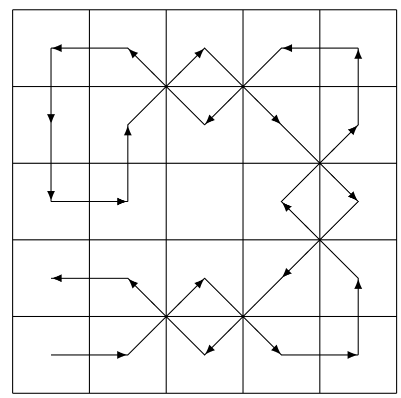
   - It can move one square at a time, horizontally, vertically, or diagonally. (These are the usual moves of the king in chess.)
   - It can move in each of the eight allowable directions at most three times in its entire route.

   The king can start at any square. Determine

   (a) whether the king can visit every square;

   (b) whether the king can visit every square except the center.

   Solution.

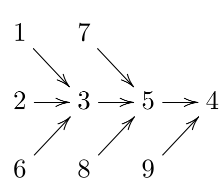
   (a) The answer is no. To visit all 25 squares, the king must make his maximum of 24 moves and thus must move in each of the eight allowable directions exactly 3 times. Three of these directions (or 9 moves) take the king from a row to the next higher row; three directions (or 9 moves) go to the next lower row, and the remaining two directions and six moves keep the king in the same row. Because the numbers of upward and downward moves are equal, the king must start and end in the same row. Similarly, he must start and end in the same column and thus in the same square. Thus the 25th square visited will repeat the first and at least one square will be missed.

   (b) The answer is yes. A solution is shown. Trickily, all known solutions are completely asymmetric.

3. Define the *digitlength* of a positive integer to be the total number of letters used in spelling its digits. For example, since "two zero one one" has a total of 13 letters, the digitlength of 2011 is 13. We begin at any positive integer and repeatedly take the digitlength. Show that after some number of steps, we must arrive at the number 4.

   Solution. We first claim that if a number has at least two digits, then the digitlength is less than the number itself. To see this, note that any digit has at most five letters; thus a 2-digit number, which is at least 10, has digitlength at most $5 \cdot 2 = 10$; a 3-digit number, which is at least 100, has digitlength at most $5 \cdot 3 = 15$; and so on. The highest possible digitlength increases at each step by 5, while the lowest possible number having a given number of digits increases by more than 5. Thus the digitlength will always be strictly less than the number unless the number and the digitlength are both 10, which does not happen since the digitlength of 10 is 7.

   When we repeatedly take the digitlength, the numbers keep decreasing as long as they have at least two digits. Since a sequence of positive integers cannot decrease forever, we must eventually hit a 1-digit number. Then, we quickly reach 4 as shown by the following diagram:

4. Suppose 2011 light bulbs are arranged in a row. Each bulb has a button under it. Pressing the button will change the state of the bulb above it (on to off or vice versa) and will also change the two neighboring bulbs, or the single neighboring bulb in the case of one of the two end buttons. Is it always possible, regardless of the initial state of the bulbs, to turn them all off by pressing some buttons?

Solution. The answer is yes.

Let us number the bulbs 1 to 2011 from left to right. Given any initial state of the bulbs, let us begin by following this algorithm: As long as at least one bulb other than bulb 1 is on, let n be the number of the rightmost lit bulb and push the button for bulb $n-1$. This will turn bulb n off and move the location of the rightmost lit bulb to the left. We may continue until either (1) all bulbs are off or (2) only bulb 1 is on. In the former case, we are done; in the latter case we then push the buttons marked $\times$:

1	2	3	4	5	6	7	8	$\cdots$	2007	2008	2009	2010	2011
$\times$		$\times$	$\times$		$\times$	$\times$			$\times$	$\times$		$\times$	$\times$

It is evident that every bulb will then change state twice, except the first, which will change state once. Thus all the bulbs will be turned off.

5. Let ABC be a triangle with $\angle ACB = 90°$. The inscribed circle of $\triangle ABC$ touches sides AC and BC at D and E, respectively. On the circumscribed circle of $\triangle ABC$, the midpoints of minor arcs AC and BC are respectively P and Q. Prove that D, E, P, and Q are all collinear.

Solution. Let M be the midpoint of AC, let O be the circumcenter of $\triangle ABC$, and let F be the point where the incircle touches AB.

Note that $\triangle CDE$ is a right isosceles triangle and therefore $\angle CDE = 45°$. Also, $\angle PMD$ is right since $OM \perp AC$ and OM passes through P. Let us prove that $\triangle PMD$ is right isosceles:

$$PM = OP - OM = \frac{AB - BC}{2} = \frac{AF + FB - BE - EC}{2} = \frac{AD - DC}{2} = \frac{AC - 2DC}{2} = MC - DC = MD.$$

Therefore $\angle PDM = 45°$ and

$$\angle PDM + \angle MDE = \angle PDM + 180° - \angle CDE = 45° + 180° - 45° = 180°.$$

This proves that P lies on line DE. Similarly, we can prove that Q lies on line DE.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that $f(0) = 2$ and for all integers x,

$$f(x+1) + f(x-1) = f(x)f(1). \tag{1}$$

Prove that for all integers x and y,

$$f(x+y) + f(x-y) = f(x)f(y). \tag{2}$$

Solution. Let us first prove the result for nonnegative y by strong induction on y. If $y = 0$, (2) becomes

$$f(x) + f(x) = f(x)f(0),$$

which is true since $f(0) = 2$, and if $y = 1$, (2) is the same as (1). Let us assume (2) for $y = z - 1$ and $y = z$ and try to prove it for $y = z + 1$. We have

$$f(x+z-1) + f(x-z+1) = f(x)f(z-1) \tag{3}$$
$$f(x+z) + f(x-z) = f(x)f(z) \tag{4}$$

Multiplying (4) by $f(1)$ and subtracting (3) yields

$$\begin{aligned} f(x+z)f(1) - f(x+z-1) + f(x-z)f(1) - f(x-z+1) &= f(x)\big[f(z)f(1) - f(z-1)\big] \\ f(x+z+1) + f(x-z-1) &= f(x)f(z+1) \end{aligned}$$

as desired.

It remains to prove (2) for $y < 0$. If $y = -z$ and $z > 0$, (2) becomes

$$f(x-z) + f(x+z) = f(x)f(-z)$$

which would follow from (2) for $y = z$ if $f(-z) = f(z)$. To prove this, let $x = 0$ and $y = z$ in (2), getting

$$\begin{aligned} f(z) + f(-z) &= f(0)f(z) \\ f(z) + f(-z) &= 2f(z) \\ f(-z) &= f(z). \end{aligned}$$

7. Let n be a positive integer which is divisible by 5 and which can be written as the sum of two (not necessarily distinct) squares. Prove that n can be written as the sum of two squares one of which is greater than or equal to four times the other.

Remark. A *square* is the square of any integer including zero.

Solution. Let $n = a^2 + b^2$, where a and b are nonnegative integers. Suppose that each of the squares a^2 and b^2 is less than 4 times the other, so

$$a < 2b \quad \text{and} \quad b < 2a.$$

Since $a^2 + b^2$ is divisible by 5, so is $a^2 + b^2 - 5b^2 = a^2 - 4b^2 = (a + 2b)(a - 2b)$. Thus 5 divides either $a + 2b$ or $a - 2b$. If $5|a - 2b$, then $5|2(a - 2b) + 5b = 2a + b$. Thus 5 divides either $a + 2b$ or $2a + b$, and by switching the labels a and b we may assume that $5|a + 2b$.

Then

$$5 \mid 4(a + 2b) - 5b = 4a + 3b$$

and

$$5 \mid 3(a + 2b) - 10b = 3a - 4b.$$

Note that

$$\begin{aligned}\left(\frac{4a+3b}{5}\right)^2 + \left(\frac{3a-4b}{5}\right)^2 &= \frac{16a^2 + 24ab + 9b^2 + 9a^2 - 24ab + 16b^2}{25} \\ &= \frac{25a^2 + 25b^2}{25} = a^2 + b^2 = n.\end{aligned}$$

We claim that

$$\left(\frac{4a+3b}{5}\right)^2 \geq 4\left(\frac{3a-4b}{5}\right)^2$$

which is equivalent to

$$4a + 3b \geq 2|3a - 4b|$$

which in turn is equivalent to the pair of inequalities

$$4a + 3b \geq 2(3a - 4b) \quad \text{and} \quad 4a + 3b \geq -2(3a - 4b).$$

Simplifying the first inequality yields $11b \geq 2a$ which is true since $2b \geq a$. Simplifying the second inequality yields $10a \geq 5b$ which is true since $2a \geq b$.