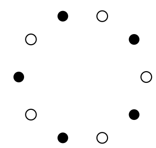
Berkeley Math Circle
Monthly Contest 5 – Solutions

1. A house has several rooms. There are also several doors, each of which connects either one room to another or a room to the outside. Suppose that every room has an even number of doors leaving it. Prove that the number of outside entrance doors is even as well.

   Solution. Every door has two "sides," one toward one room and one toward either another room or the outside. Clearly the total number of sides, being twice the number of doors, is even. However, for each room, the number of sides pointing to it is even. Since even subtracted from even gives even, the number of sides pointing to the outside is also even.

2. 

   As shown in the diagram above, the vertices of a regular decagon are colored alternately black and white. We would like to draw colored line segments between the points in such a way that

   (a) Every black point is connected to every white point by a line segment.

   (b) No two line segments of the same color intersect, except at their endpoints.

   Determine, with proof, the minimum number of colors needed.

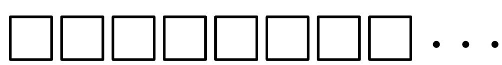
   Solution. Note that the five main diagonals of the decagon must all be different colors since they all intersect at the decagon's center. Thus, at least 5 colors are needed. To see that 5 colors are also sufficient, we can simply assign each black point a color and use that color to connect it with all the white points.

3. Two congruent line segments AB and CD intersect at a point E. The perpendicular bisectors of AC and BD intersect at a point F in the interior of $\angle AEC$. Prove that EF bisects $\angle AEC$.

   Solution. Since F is on the perpendicular bisectors of AC and BD, $FA = FC$ and $FB = FD$. Also $AB = CD$ is given. Thus $\triangle FAB \cong \triangle FCD$ by SSS. Since corresponding heights of congruent triangles and equal, F is equidistant from AB and CD, implying that F is on the bisector of $\angle AEC$.

   Remark. The problem was intended to read that F lies in the interior of $\angle AED$. Some contestants pointed out that the condition given—that F lies in the interior of $\angle AEC$—implies a stronger result, namely that $AE = EC$ and therefore the perpendicular bisectors of AC and BD coincide. This does not, however, impact the truth of the problem as stated.

4. We have a row of boxes that is infinite in one direction, as shown.

   Determine if it is possible to fill each box with a positive integer such that the number in every box (except the leftmost one) is greater than the average of the numbers in the two neighboring boxes.

   Solution. The answer is no.

   Denote the numbers in the boxes by $a_0, a_1, a_2, \ldots$. Then, if the conditions are satisfied, we have for all $n \geq 1$

$$a_n > \frac{a_{n-1} + a_{n+1}}{2}$$
$$2a_n > a_{n-1} + a_{n+1}$$
$$a_n - a_{n-1} > a_{n+1} - a_n.$$

   This says that the differences $a_n - a_{n-1}$ form a strictly decreasing sequences. Since the differences are all integers, they must eventually become negative.

   At this point the sequence a_n is itself strictly decreasing, so by the same token, eventually the a_n's become negative, contradicting the condition that they are all positive.

5. Given a triangle ABC, we draw three circles with respective diameters AB, BC, and CA. Prove that there exists a point that is inside all three circles.

   Solution. We claim that the center I of the triangle's inscribed circle is such a point. To see this, note that

$$\angle BIC = 180 - \angle ICB + \angle CBI = 180 - \frac{\angle C + \angle B}{2} = 180 - \frac{180 - A}{2} = 90 + \frac{A}{2} > 90.$$

   Thus $\triangle BIC$ is obtuse and if we drop a perpendicular BD from B to CI, then D lies on the extension of ray CI. The circle with diameter BC passes through D since $\angle BDC = 90$, and thus I, an interior point of chord CD, lies inside the circle. By symmetry, we can conclude that I lies inside the other two circles as well.

6. At a certain school, there are 6 subjects offered, and a student can take any combination of them. It is noticed that for any two subjects, there are fewer than 5 students taking both of them and fewer than 5 students taking neither. Determine the maximum possible number of students at the school.

   Solution. Let us count in two ways the number of ordered pairs $(s, \{c_1, c_2\})$, where s is a student, $\{c_1, c_2\}$ is an unordered pair of two distinct courses, and s is either taking both c_1 and c_2 or neither c_1 nor c_2.

   First, we consider any of the $\binom{6}{2} = 15$ pairs $\{c_1, c_2\}$ of distinct courses. By the given conditions, at most 4 students are taking neither c_1 nor c_2 and at most 4 take both, implying that $\{c_1, c_2\}$ gives rise to at most 8 pairs $(s, \{c_1, c_2\})$, so the total number of such pairs is at most 120.

   Second, let N be the number of students. Consider any student s. If k denotes the number of courses s is taking, then the number of pairs of courses $\{c_1, c_2\}$ such that s is taking both or neither of them is

$$\binom{k}{2} + \binom{6-k}{2}$$

   Plugging in $k = 0, 1, \ldots, 6$ shows that the minimum value of this expression is 6, attained at $k = 3$. Thus there are at least $6N$ pairs $(s, \{c_1, c_2\})$.

   Putting this together, we see that $6N \leq 120$, so N is at most 20. A total of 20 students is indeed possible, as can be seen by letting each student take a distinct triplet of courses from the $\binom{6}{3} = 20$ possible triplets.

7. Let a and b be real numbers. Prove that the polynomial

$$P(x) = x^3 + (2a+1)x^2 + (2a^2 + 2a - 3)x + b$$

   does not have three distinct rational roots.

   Solution. Assume the contrary, and suppose that the polynomial has rational roots u, v, w. Then by Vi'ete's formulas,

$$u + v + w = -(2a+1)$$
$$uv + vw + wu = 2a^2 - 2a - 3$$

   Squaring the first equation and subtracting twice the second yields

$$(u+v+w)^2 - 2uv - 2vw - 2wu = (2a+1)^2 - (4a^2 + 4a - 6)$$
$$u^2 + v^2 + w^2 = 4a^2 + 4a + 1 - 4a^2 - 4a + 6 = 7.$$

   We claim that there are no rational numbers u, v, w satisfying $u^2 + v^2 + w^2 = 7$. Assuming that there were, let s be their least common denominator, so $u = p/s$, $v = q/s$, $w = r/s$ for integers p, q, r, s with GCD 1. Then

$$\frac{p^2 + q^2 + r^2}{s^2} = 7$$
$$p^2 + q^2 + r^2 = 7s^2.$$

   If we work mod 8, this equation assumes the symmetrical form

$$p^2 + q^2 + r^2 \equiv (-1)s^2$$
$$p^2 + q^2 + r^2 + s^2 \equiv 0.$$

   Using the fact that the squares mod 8 are 0, 1, and 4, it is not hard to conclude that p, q, r, and s must all be even, contradicting the assumption that their GCD is 1.