Berkeley Math Circle
Monthly Contest 5
Due February 8, 2011

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 5, Problem 3
by Bart Simpson
in grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

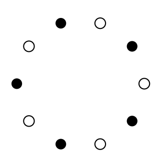
Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A house has several rooms. There are also several doors, each of which connects either one room to another or a room to the outside. Suppose that every room has an even number of doors leaving it. Prove that the number of outside entrance doors is even as well.

2. 

   As shown in the diagram above, the vertices of a regular decagon are colored alternately black and white. We would like to draw colored line segments between the points in such a way that

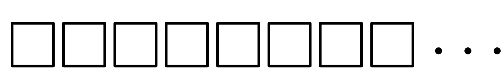
   (a) Every black point is connected to every white point by a line segment.

   (b) No two line segments of the same color intersect, except at their endpoints.

   Determine, with proof, the minimum number of colors needed.

3. Two congruent line segments AB and CD intersect at a point E. The perpendicular bisectors of AC and BD intersect at a point F in the interior of $\angle AEC$. Prove that EF bisects $\angle AEC$.

4. We have a row of boxes that is infinite in one direction, as shown.

   Determine if it is possible to fill each box with a positive integer such that the number in every box (except the leftmost one) is greater than the average of the numbers in the two neighboring boxes.

5. Given a triangle ABC, we draw three circles with respective diameters AB, BC, and CA. Prove that there exists a point that is inside all three circles.

6. At a certain school, there are 6 subjects offered, and a student can take any combination of them. It is noticed that for any two subjects, there are fewer than 5 students taking both of them and fewer than 5 students taking neither. Determine the maximum possible number of students at the school.

7. Let a and b be real numbers. Prove that the polynomial

$$P(x) = x^3 + (2a+1)x^2 + (2a^2 + 2a - 3)x + b$$

does not have three distinct rational roots.