Berkeley Math Circle
Monthly Contest 3 – Solutions

1. You are given an $m \times n$ chocolate bar divided into 1×1 squares. You can break a piece of chocolate by splitting it into two pieces along a straight line that does not cut through any of the 1×1 squares. What is the minimum number of times you have to break the bar in order to separate all the 1×1 squares?

   Solution. We note that the number of separate pieces of chocolate increases by 1 at each cut. We begin with 1 piece and end with mn pieces, so we must make $mn - 1$ cuts. Thus $mn - 1$ is the minimum (and also the maximum) number of cuts necessary to separate all the 1×1 squares.

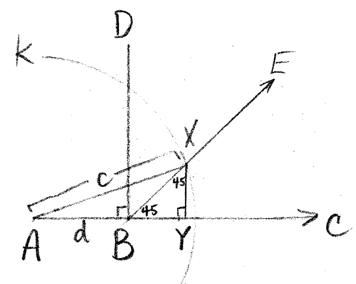
2. Let n be a positive integer. Prove that the nth prime number is greater than or equal to $2n - 1$.

   Solution. We can verify that the first prime, 2, is greater than $2 \cdot 1 - 1 = 1$ and the second prime, 3, is equal to $2 \cdot 2 - 1$. From then, on, since all primes except 2 are odd, the difference between consecutive primes is at least 2. Therefore, for $n \geq 3$,

$$\begin{aligned} n\text{th prime} &\geq 3 + \overbrace{2 + 2 + \cdots + 2}^{n-2 \text{ differences}} \\ &= 3 + 2(n-2) \\ &= 3 + 2n - 4 \\ &= 2n - 1. \end{aligned}$$

3. Given the hypotenuse and the difference of the two legs of a right triangle, show how to reconstruct the triangle with ruler and compass.

   Solution. Here is one construction. Let $AB = d$ be the segment representing the difference of the two legs. Extend AB to C and raise a perpendicular BD to AC at B. Bisect angle DBC to make ray BE with $\angle EBC = 45°$. Now set the compass to the length c of the hypotenuse and draw a circle k of radius c centered at A. Because the hypotenuse of a right triangle is longer than the difference of the legs, $c > AB$, and thus k will intersect ray BE at a point X. Drop a perpendicular XY from X to AC. Then $\triangle AXY$ is a right triangle with hypotenuse AX, and the difference $AY - XY$ of the legs equals $AY - BY = AB$ since BXY is an isosceles right triangle.



   If there were any other right triangle satisfying the same specifications, we could put it in the position $AX'Y'$ with the smallest angle at A, the longer leg AY' on ray AC, and X' above Y'. Then X' would lie on k because $AX' = c$; also, since $AX' - X'Y' = AB = AY' - BY'$, $BX'Y'$ is an isosceles right triangle, and so X' lies on $\overrightarrow{BE}$. However, since B is inside k, k and $\overrightarrow{BE}$ can only intersect once. Thus $X' = X$ and $Y' = Y$.

4. Show that each number in the sequence

$$49, \quad 4489, \quad 444889, \quad 44448889, \quad \ldots$$

   is a perfect square.

   Solution. Let x_n denote the nth number in the sequence. If we multiply x_n by 9 by the standard method, we see that the calculations $9 \cdot 8 + 8 = 80$ and $9 \cdot 4 + 4 = 40$ occur repeatedly:

$$\begin{array}{rrr} & \overbrace{44\cdots44}^{n \text{ digits}} & \overbrace{88\cdots89}^{n \text{ digits}} \\ \times & & 9 \\ \hline 4 & 00\cdots04 & 00\cdots01 \end{array}$$

   The result is $9x_n = 4 \cdot 10^{2n} + 4 \cdot 10^n + 1 = (2 \cdot 10^n + 1)^2$, so

$$x_n = \left(\frac{2 \cdot 10^n + 1}{3}\right)^2.$$

   To see that the expression in parentheses is an integer, we may note that the sum of the digits of the numerator $200\cdots001$ is 3, a multiple of 3.

5. Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers such that for each $n \geq 1$,

$$a_{n+2} = a_{n+1} + a_n.$$

Prove that for all $n \geq 2$, the quantity

$$|a_n^2 - a_{n-1}a_{n+1}|$$

does not depend on n.

Solution. It suffices to prove that increasing n to $n+1$ does not change the value, i.e. that for $n \geq 2$,

$$|a_n^2 - a_{n-1}a_{n+1}| = |a_{n+1}^2 - a_n a_{n+2}|.$$

We will prove more specifically that

$$a_n^2 - a_{n-1}a_{n+1} = -(a_{n+1}^2 - a_n a_{n+2})$$

by transforming:

$$\begin{aligned} a_n^2 - a_{n-1}a_{n+1} &\stackrel{?}{=} -(a_{n+1}^2 - a_n a_{n+2}) \\ a_n^2 - a_{n-1}a_{n+1} &\stackrel{?}{=} -a_{n+1}^2 + a_n a_{n+2} \\ a_{n+1}^2 - a_{n-1}a_{n+1} &\stackrel{?}{=} a_n a_{n+2} - a_n^2 \\ a_{n+1}(a_{n+1} - a_{n-1}) &\stackrel{?}{=} a_n(a_{n+2} - a_n). \end{aligned}$$

Using the given relation $a_{n+2} = a_{n+1} + a_n$, we see that the right side equals $a_n \cdot a_{n+1}$. Replacing n by $n-1$ in the given relation gives $a_{n+1} = a_n + a_{n-1}$, so the left side equals $a_{n+1} \cdot a_n$ and thus the equality is true.

6. The inscribed circle of a triangle ABC touches the sides BC, CA, AB at D, E, and F respectively. Let X, Y, and Z be the incenters of triangles AEF, BFD, and CDE, respectively. Prove that DX, EY, and CZ meet at one point.

Solution. Consider the midpoint M of arc EF on the incircle of $\triangle ABC$. Angles AFM and MFE are equal since they intercept equal arcs FM and ME, and so M is on the bisector of $\angle AFE$. Similarly, M is on the bisector of $\angle FEA$, and therefore M coincides with X. Moreover, angles FDX and XDE are equal since they intercept equal arcs FX and XE, and so DX is the angle bisector of $\angle D$ in $\triangle DEF$. Similarly, EY and FZ are the other two angle bisectors in $\triangle DEF$. But the three angle bisectors in a triangle always meet!

7. Define a sequence $a_0, a_1, a_2, \ldots$ in the following way: $a_0 = 0$, and for $n \geq 0$,

$$a_{n+1} = a_n + 5^{a_n}.$$

Let k be any positive integer. Prove that the remainders when $a_0, a_1, \ldots, a_{2^{k-1}}$ are divided by 2^k are all different.

Remark. It was intended to prove that $a_0, a_1, \ldots, a_{2^k-1}$ (not just up to $a_{2^{k-1}}$) have different remainders. This typo does not affect the truth of the problem, and we will prove the strengthened statement.

Solution. We begin with a simple numerical lemma.

Lemma 1. *For all $k \geq 0$, $5^{2^k} - 1$ is divisible by 2^{k+2}.*

Proof. By induction. For $k = 0$, the statement may be checked directly. To step from k to $k+1$, we write

$$5^{2^{k+1}} - 1 = \left(5^{2^k}\right)^2 - 1 = \left(5^{2^k} - 1\right)\left(5^{2^k} + 1\right)$$

and note that the first factor is divisible by 2^{k+2} by the induction hypothesis and the second factor is clearly divisible by 2, so the product is divisible by 2^{k+3}. □

Now we turn to the main lemma of the proof.

Lemma 2. *Fix $k \geq 0$. For all $r \geq 0$, the difference*

$$a_{r+2^k} - a_r$$

is divisible by 2^k, not divisible by 2^{k+1}, and independent of r mod 2^{k+2}.

Proof. By induction. For $k = 0$,

$$a_{r+1} - a_r = 5^{a_r}$$

is clearly divisible by 1, not divisible by 2, and congruent to the constant value $1^{a_r} = 1 \bmod 4$. Now assume that the statement is true for k; we will prove it for $k+1$. We know that $a_{r+2^k} - a_r$ has the form $2^k \cdot m$ for m odd. The difference $a_{r+2^{k+1}} - a_{r+2^k}$ has the same value, $2^k \cdot m$, to the modulus 2^{k+2}. Therefore, mod 2^{k+2},

$$a_{r+2^{k+1}} - a_r = (a_{r+2^{k+1}} - a_{r+2^k}) + (a_{r+2^k} - a_r) \equiv 2^k \cdot m + 2^k \cdot m = 2^{k+1}m,$$

so this difference is divisible by 2^{k+1} but not 2^{k+2}.

It remains to prove that $a_{r+2^{k+1}} - a_r$ is constant mod 2^{k+3}, i.e. that 2^{k+3} divides the difference between two values for consecutive values of r:

$$\begin{aligned}(a_{r+2^{k+1}+1} - a_{r+1}) - (a_{r+2^{k+1}} - a_r) &= (a_{r+2^{k+1}+1} - a_{r+2^{k+1}}) - (a_{r+1} - a_r) \\ &= 5^{a_{r+2^{k+1}}} - 5^{a_r} \\ &= 5^{a_r}(5^{a_{r+2^{k+1}} - a_r} - 1).\end{aligned}$$

We have already proved that $a_{r+2^{k+1}} - a_r$ is divisible by 2^{k+1}. By Lemma 1, $5^{2^{k+1}} \equiv 1 \mod 2^{k+3}$, so $5^{a_{r+2^{k+1}} - a_r}$ is also 1 $\mod 2^{k+3}$, completing the proof. □

To solve the problem, suppose that r and s are two nonnegative integers such that $0 \leq r < s \leq 2^k - 1$ and $a_r \equiv a_s \mod 2^k$. Let $s - r = 2^\ell \cdot m$, where m is odd. Mod $2^{\ell+1}$, $a_{t+2^\ell} - a_t \equiv 2^\ell$ for all $t \geq 0$ and therefore

$$a_s - a_r \equiv (a_{r+2^\ell} - a_r) + (a_{r+2\cdot 2^\ell} - a_{r+2^\ell}) + \cdots + (a_{r+m\cdot 2^\ell} - a_{r+(m-1)2^\ell}) \equiv m \cdot 2^\ell \not\equiv 0,$$

implying that $\ell + 1 \geq k + 1$, i.e. $\ell \geq k$. Thus $s - r \geq 2^k$, which is a contradiction.