Berkeley Math Circle
Monthly Contest 1
Due October 5, 2010

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Are there natural numbers a and b that make the equation

$$2a^2 + 1 = 4b^2$$

true? (The natural numbers are the counting numbers, $1, 2, 3, \ldots$.)

Remark. If you think that the answer is *yes*, you should give the numbers a and b and show that they make the equation true. If you think that the answer is *no*, you should explain why ALL choices of a and b make the equation false.

2. Given three points A, B, C known to lie on a circle, prove that one can reconstruct the original circle with a straightedge and compass.

Remark. To get the full 7 points, your solution must include the following:

(a) A careful description (in words!) of a method that begins with three points A, B, C and ends up with some circle.

(b) An explanation of why the circle that you have found must go through all three points.

(c) Why you have found the RIGHT circle, that is, why there can be no OTHER circles passing through the same three points.

3. (a) You are given the expression

$$1 \diamond 2 \diamond 3 \diamond 4 \diamond 5 \diamond 6 \diamond 7.$$

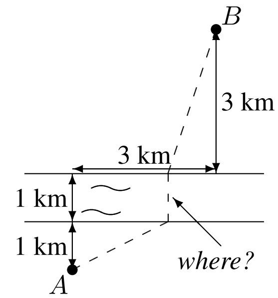
Determine whether it is possible to replace one of the symbols $\diamond$ with $=$ and the other symbols $\diamond$ with $+$ or $-$ so as to end up with a correct equation. If this is possible, show your example and demonstrate that it indeed works. If this is impossible, explain rigorously why it is impossible.

(b) The same question for the expression

$$1 \diamond 2 \diamond 3 \diamond 4 \diamond 5 \diamond 6.$$

4. Two villages, A and B, lie on opposite sides of a straight river in the positions shown. There will be a market in village B, and residents of village A wish to attend. The people of village A would like to build a bridge across and perpendicular to the river so that the total route walked by residents of A to the bridge, across the bridge, and onward to B is as short as possible. How can they find the exact location of the bridge, and how long will the route be?

   Remark. Explain rigorously your solution: where and how to find the bridge location, and why the resulting total route from A to B will be the minimal possible of all such routes.



5. We have four bowls labeled A, B, C, D in a row, and we have four indistinguishable marbles which we can initially distribute among the bowls any way we like. A *move* consists of transferring one marble from a bowl to one of the adjacent bowls in the row. Is it possible to perform a succession of moves in which every distribution of the marbles appears exactly once?

6. In the interior of a triangle ABC with area 1, points D, E, and F are chosen such that D is the midpoint of AE, E is the midpoint of BF, and F is the midpoint of CD. Find the area of triangle DEF.

7. Let a, b, and c be positive real numbers such that
$$abc = 1$$
   and
$$a + b + c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$
   Prove that for every positive integer n,
$$a^n + b^n + c^n > \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}.$$