On p-adic numbers

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p-adic distance

Let p be a prime number. Every rational number can be written in the form $r = \frac{k}{l} p^n$, where n, k, l are integers, k and l are relatively prime and p does not divide k and l. Define a *p-adic norm* as

$$|r|_p = \frac{1}{p^n}, \text{ if } r \neq 0, \text{ and } |0|_p = 0.$$

For example $|20|_2 = \frac{1}{4}$, $|\frac{3}{40}|_2 = 8$.

1. Check the following properties of p-adic norm

(a) $|ab|_p = |a|_p |b|_p$;

(b) if $|a|_p > |b|_p$, then $|a+b|_p = |a|_p$;

(c) if $|a|_p = |b|_p$, then $|a+b|_p < |a|_p$;

(d) $|a+b|_p < |a|_p + |b_p|$.

A *distance* on a set S is a function $d(a,b)$ satisfying the following properties

if $a \neq b$, then $d(a,b) > 0$, if $a = b$, then $d(a,b) = 0$;

$d(a,b) = d(b,a)$ (symmetry);

$d(a,c) \leq d(a,b) + d(b,c)$.

2. Check that the following functions on the plane satisfy the distance axioms

(a)$d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$;

(b)$d((a_1,a_2),(b_1,b_2)) = |a_1-b_1| + |a_2-b_2|$;

(c)$d((a_1,a_2),(b_1,b_2)) = \max(|a_1-b_1|, |a_2-b_2|)$.

A circle with center a and the radius r is the set of all points x such that $d(a,x) \leq r$.

3. Draw the circles with radius 1 and center at the origin for distance functions in the previous problem.

Define the *p-adic distance* on the set of rational numbers $\mathbb{Q}$ by

$$d_p(a,b) = |a-b|_p.$$

4. Check that d_p satisfies the distance axioms.

5. Show that for any three rational numbers a, b, c at least two of three distances $d_p(a, b)$, $d_p(b, c)$ and $d_p(a, c)$ are equal, i.e. every triangle in p-adic distance is isosceles.

6. Show that any two p-adic circles either do not intersect or lie one inside the other.

Triangulation of a square

We will use 2-adic norm to attack the following problem.

7. It is impossible to divide a square into odd number of triangles of equal area.

Assume that the vertices of a square have coordinates $(0, 0), (1, 0), (0, 1), (1, 1)$ and all vertices of the triangles have rational coordinates.

Let us color all vertices using three according to the following the following rule. Suppose a point has coordinates (x, y)

Color the point green if $|x|_2, |y|_2 < 1$;

Color the point red if $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$;

Color the point blue if $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

In this way the vertices of the square are colored green, red, blue, red.

8. Prove that all vertices lying on the same line are colored by two of three colors.

9. Show that one can find a triangle whose vertices are colored with three different colors.

10. Given three points $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$. Check that the area of a triangle ABC equals

$$\frac{|x_1 y_2 - y_1 x_2|}{2},$$

where $x_1 = b_1 - a_1, x_2 = b_2 - a_2, y_1 = c_1 - a_1, y_2 = c_2 - a_2$.

11. Let ABC be a triangle whose vertices are colored in three different colors. Show that the 2-adic norm of its area is less than.

12. Solve the problem 7 in case when the vertices of all triangles have rational coordinates.

p-adic numbers

13. Show that every positive integer can be written uniquely in the form $a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$ where $0 \leq a_i \leq p - 1$.

14. Show that every positive rational number of the form $\frac{m}{p^n}$ can be written uniquely as the sum

$$a_{-s}p^{-s} + \cdots + a_{-1}p^{-1} + a_0 + a_1p + a_2p^2 + \cdots + a_kp^k$$

where $0 \le a_i \le p-1$.

A p-adic number is an expression of the form

$$a_{-s}p^{-s} + \cdots + a_{-1}p^{-1} + a_0 + a_1p + a_2p^2 + \ldots,$$

with $0 \le a_i \le p-1$ where we allow infinitely many positive powers of p. It is usually written as

$$\ldots a_0.a_{-1}a_{-2}\ldots a_{-s}$$

with finitely many p-adic digits on the right of the dot and maybe infinitely many p-adic digits on the left of the dot.

For example,

$$\cdots 11111.1$$

is a 2-adic number with infinitely many 1-s on the left. Look at a finite piece of this number

$$r_n = 1 + 2 + 2^2 + \cdots + 2^n = \frac{2^{n+1}-1}{2-1} = -1 + 2^{n+1}.$$

It is clear that the bigger n is the smaller is the 2-adic distance between -1 is r_n. Thus, -1 can be approximated by r_n in 2-adic distance. Thus, we have

$$-1 = \cdots 1111.1$$

in 2-adic form. Denote by $\mathbb{Q}_p$ the set of p-adic numbers.

15. Define addition and multiplication on the set of p-adic numbers and check that the usual laws hold for them, for example $a(b+c) = ab + ac$ e.t.c.

16. Show that for every non-zero x in $\mathbb{Q}_p$ there exist y and z such that $x + y = 0$ and $xz = 1$. Hence one can subtract and divide p-adic numbers.

17. Check that the 2-adic form of $\frac{1}{3}$ is

$$\cdots 10101.1$$

Write $\frac{1}{7}$ in 2-adic form.

18. Show that every rational number can be approximated by finite p-adic numbers in p-adic distance, and the corresponding infinite p-adic number is periodic.

19. Show that $\mathbb{Q}_2$,$\mathbb{Q}_3$ and $\mathbb{Q}_5$ do not contain $\sqrt{2}$, but $\mathbb{Q}_7$ contains it. In other words, the equation $x^2 = 2$ has a solution in 7-adic numbers.

20. If p is odd, a be an integer relatively prime to p. Show that if a is a perfect square modulo p, then a is a perfect square in $\mathbb{Q}_p$.

21. Define p-adic norm and p-adic distance in $\mathbb{Q}_p$.

22. Solve Problem 7 in case when the vertices of all triangles have 2-adic coordinates.

In fact, it is possible to define 2-adic norm for all real numbers, although it is not very easy.

Reference On 2-adic numbers. Bekker, Ionin, Vostokov. Kvant 1979.