Geometric Transformations with an Introduction to Techniques from Projective Geometry
Part I

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Questions

Suppose a Quarter coin rolls, without slipping, around the edge of a second Quarter. What is the locus of a single point on the edge of the rolling quarter?

Suppose a Penny touches a Nickel, which touches a Dime, which touches a Quarter, which touches the original Penny. Show that the four points of contact between the coins all lie on a circle.

Given a planet with orbital radius 1AU and another with orbital radius 2AU, suppose the first has period one year, and the second has period two years (e.g. approximately the situation with Earth and Mars).
What does the orbit of the second planet look like from the (moving) location of the first?

Exercises

1. Given a circle, construct a circle orthogonal to it.

2. Given two circles non-intersecting circles, construct a circle orthogonal to both.

3. Prove that any three circles can be inverted into three circles whose centers are collinear.

4. Prove that $A'B' = r^2 \cdot \dfrac{AB}{OA \cdot OB}$ where O is the center of inversion, and r is the radius of circle O.

5. Prove that orthogonal circles remain fixed under inversion by each other.

6. If two circles are orthogonal, then the diameter of one circle cuts the other in a pair of points which are inverses of one another. Moreover, this is true of any radial line of one circle which cuts the other circle in two points.

7. Prove that the center of a circle does not invert to the center of its image.

8. Find the polar coordinate equation for a parabola; for an ellipse; for a hyperbola.

9. Prove that there is "only one" parabola, i.e. under scaling all parabolas are congruent.

10. ***What is the image of a parabola under inversion through its focus?***

11. Given two circles, find a point from which the four tangent segments to the two circles are all congruent; what is the locus of all such points?

12. *Prove that a pair of disjoint circles can be inverted into a pair of concentric circles.*

13. Prove that the centers of the circles in a Steiner Chain all lie on an ellipse.

14. Given three concurrent circles, how many circles may be constructed tangent to all three?

15. Prove that any three disjoint circles can be inverted into circles whose centers are collinear.

Write new theorems by inverting the given set-up

1. If the opposite angles of a quadrilateral are supplementary, then the quadrilateral may be inscribed in a circle (converse is trivial).

2. The line joining the centers of two intersecting circles is perpendicular to their common chord.

3. The altitudes of a triangle are concurrent.

Suggested Reading

Excursions in Geometry, C. Stanley Ogivly (Dover Books)

Introduction to Geometry, Coxeter (Wiley and Sons)

Geometry: A Comprehensive Course, Dan Pedoe (Dover Books)

Reflections on the Arbelos, Harold P. Boas, American Mathematical Monthly **113** (March 2003)
(http://www.math.tamu.edu/harold.boas/preprints/arbelos.pdf)

Modern Geometries, James Smart (Brooks/Cole)

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