As a warm-up, here are some classic problems that you may have seen before:

1. Let ABC be an equilateral triangle. Let P be a point in ABC such that the distances from P to A, B and C are 3, 4 and 5, respectively. Find the area of $\triangle ABC$.

2. Let ABC be an equilateral triangle. Let P be a point on arc BC of the circumcircle of ABC. Show $PA = PB + PC$.

   These two problems are in fact remarkably connected. Let's now analyze the situation more closely.

3. Show that for any other point P, $PA > PB + PC$. This suggests that we can construct a triangle that has side lengths PA, PB, PC for any point P.

4. In fact, using a rotation that's the reverse of problem 1, we can solve problems 2 and 3. Note that when P is on the circumcircle, we get a degenerate triangle.

5. Rather than using Ptolemy's theorem to solve problems 2 and 3, we can use 4 to prove Ptolemy's theorem.

6. We are now armed to tackle a question proposed by Fermat as a challenge to Evangelista Torricelli. Given a triangle, none of whose angles exceeds $\frac{2\pi}{3}$, find the point F in the plane that minimizes $FA + FB + FC$.

7. *What happens when one of the angles exceeds $\frac{2\pi}{3}$?

8. Given a triangle ABC, construct an equilateral triangle $A'BC$ externally on side BC. Construct B' and C' similarly. Show that $AA' = BB' = CC'$, and that the lines AA', BB', CC' concur. Moreover, show that these lines make angles of $\frac{\pi}{3}$ with one another.

9. (USAMO 1974/5) Consider two triangles $\triangle ABC$ and $\triangle PQR$. Let D be a point inside $\triangle ABC$ such that $\angle ADB = \angle BDC = \angle CDA = \frac{2\pi}{3}$ and let $DA = u$, $DB = v$ and $DC = w$. Let M be a point inside an equilateral $\triangle PQR$ with side length x such that $MP = a$, $MQ = b$ and $MR = c$, where a, b and c are the sides of $\triangle ABC$. Show $u + v + w = x$.

10. *There exists one other equilateral triangle $\triangle P'Q'R'$ such that there is a point M' such that $M'P' = a$, $M'Q' = b$ and $M'R' = c$. Can you find it?

11. *Find the analogue of problem 9 for this equilateral triangle $\triangle P'Q'R'$. (You will also find another "Fermat" point F'.)

12. Erect equilateral triangles externally on the sides of $\triangle ABC$. Let their centers be O_1, O_2 and O_3. Show that $\triangle O_1O_2O_3$ is equilateral. (One approach: O_2AO_3F is a kite)

13. Erect equilateral triangles internally on the sides of $\triangle ABC$. Let their centers be N_1, N_2 and N_3. Show that $\triangle N_1N_2N_3$ is equilateral, but with opposite orientation to $\triangle O_1O_2O_3$.

    So here's the punchline:

14. Show that $\triangle PQR$ in problem 9 and $\triangle O_1O_2O_3$ in problem 12 are related by a factor of $\frac{1}{\sqrt{3}}$. In fact F, O_1, O_2, O_3 are in a similar configuration to M, P, Q, R. Similarly, show that $\triangle P'Q'R'$ from problem 10 and $\triangle N_1N_2N_3$ from problem 13 are related by a factor of $\frac{1}{\sqrt{3}}$.

15. Show furthermore that $[PQR] - [P'Q'R'] = 3[ABC]$ or that, using signed areas, $[O_1O_2O_3] + [N_1N_2N_3] = [ABC]$.

16. Show that $\triangle O_1O_2O_3$ and $\triangle N_1N_2N_3$ have the same center. Their center is the centroid of ABC.

    We've discovered a lot! Let's summarize what we know. Given a triangle ABC with side lengths a, b and c, we can construct an equilateral triangle PQR and a point M such that $MP = a$, $MQ = b$ and $MR = c$. In fact, there are two such triangles an "outer" one (problem 9) and an "inner" one (problem 10). These are related to the outer and inner Napoleon triangles. They are also related to the Fermat point.

    Let's rephrase this in another way. Given an equilateral triangle PQR, and some triangle ABC with side lengths a, b and c, we can construct a point M such that $MP : MQ : MR = a : b : c$.

17. When ABC is a degenerate triangle, there is exactly one such point, and when ABC is nondegenerate there are two such points. They turn out to be inverses of one another with respect to the circumcircle of PQR.

    This might lead us to consider the following problem: given three points in the plane, P, Q, and R, and three lengths x, y and z, when can we construct a point M such that $MP : MQ : MR = x : y : z$? Certainly Ptolemy's inequality implies that we need $QR \cdot x + RP \cdot y \geq PQ \cdot z$. Is this sufficient? It turns out it is.

18. **Suppose $QR \cdot x$, $RP \cdot y$ and $PQ \cdot z$ satisfy the triangle inequality. Then we can construct a point M such that $MP : MQ : MR = x : y : z$. We proceed as follows. Construct a triangle ABC with side lengths a, b and c proportional to $QR \cdot x$, $RP \cdot y$ and $PQ \cdot z$. Construct triangles $A'BC$, $AB'C$ and ABC' all externally to ABC so that $A'BC$, $AB'C$, and ABC' are all similar to PQR. Then AA', BB', CC' concur in a point F which also lies on the circumcircles of triangles $A'BC$, $AB'C$ and ABC'. This point F minimizes $QR \cdot AF + RP \cdot BF + PQ \cdot CF$.

19. **Let O_1, O_2 and O_3 be the circumcenters of $A'BC$, $AB'C$ and ABC'. Then $O_1O_2O_3$ is similar to PQR. Moreover, $FO_1 : FO_2 : FO_3 = x : y : z$. Had we constructed $A'BC$, $AB'C$ and ABC' internally, we would have found another triangle $N_1N_2N_3$ similar to PQR but with opposite orientation. Using signed areas, $[O_1O_2O_3] + [N_1N_2N_3] = [ABC]$.

20. **Let us rephrase what we know. Given any triangle PQR, we can construct a point M with $MP : MQ : MR = x : y : z$ whenever $QR \cdot x$, $RP \cdot y$ and $PQ \cdot z$ satisfy the triangle inequality. When this triangle is degenerate, we can construct exactly one such point. When this triangle is nondegenerate, we can can construct two such points. They turn out to be inverses of one another with respect to the circumcircle of PQR.

    Now given PQR and a point M in the plane, how can we construct a triangle ABC with side lengths proportional to $QR \cdot PM$, $RP \cdot QM$, $PQ \cdot RM$? One way comes from problem 1: we construct a point S with PSQ similar to PMR. Then QSM is the desired triangle. (Note that this was also how we proved Ptolemy's theorem.)

    Another way to proceed is to drop perpendiculars from M to QR, RP and PQ and let the feet of these perpendiculars be A, B and C. This will lead us to our next topic: the Miquel point.

21. Show that $BC = PM \cdot QR/2r$, where r is the radius of the circumcircle of PQR. Conclude that ABC is the desired triangle. (Observe that we could have concluded this from the fact that O_2AO_3F is a kite used in problem 12.)

22. Use this to give yet another proof of Ptolemy's inequality. When does equality hold?

23. (Simson Line) The feet of the perpendiculars from a point P to the sides of a triangle ABC are collinear if and only if P is on the circumcircle of ABC.

24. (Miquel point) Let P, Q, R be points on sides BC, CA, AB of a triangle ABC. Then the circumcircles of AQR, PBR and PQC concur in a point, the Miquel point M associated to the triple P, Q, R.

25. The lines from the Miquel point M to the points P, Q, R make equal angles with the sides BC, CA, AB.

26. $\angle BMC = \angle BAC + \angle QPR$.

27. Analyze all triples which have the same Miquel point M. They form a system of similar triangles obtained by spiral similarity about M.

28. *Show that we can make PQR similar to any given triangle we like.

29. *Show that there are two points M, M' which yield systems of triangles similar to PQR. The two sets of triangles will have opposite orientations. Moreover, M and M' are inverses with respect to the circumcircle of ABC. Compare to problems 17 and 20.

    We can quickly deduce the following facts. 1) The centers of any set of Miquel circles are vertices of a triangle similar to ABC. 2) Two or more directly similar triangles with corresponding vertices on the same sides of ABC have the same Miquel point. 3) If three circles concur in a point, it is possible to start at any point of one of the circumferences ad draw a triangle whose vertices lies on the circles and whose sides pass through the corresponding intersections; all such triangles are similar. Note that problem 21 gives us a relationship between the side lengths of ABC, PQR, the circumradius of ABC and the lengths AM, BM, CM.

30. Problem 26 allows us to conclude that if P, Q, R are collinear, then M lies on the circumcircle of ABC. This is related to the Simson line.

31. Analyze the Simson lines of the vertices A, B, and C and also the points diametrically opposite them.

32. Show that the circumcircles of the four triangles determined by four lines are concurrent.

33. Given four lines, there is exactly one point from which the perpendiculars from this point to the four lines are collinear.

34. *Prove that the centers of the the four circumcircles in problem 32 lie on a circle with which also passes through the common point of the four circumcircles.

35. (TST 2008) *Let P, Q, and R be the points on sides BC, CA, and AB of an acute triangle ABC such that triangle PQR is equilateral and has minimal area among all such equilateral triangles. Prove that the perpendiculars from A to line QR, from B to line RP, and from C to line PQ are concurrent. These perpendiculars concur in the Fermat point.

36. **Find an analogous statement to the previous problem for the other Fermat point. Also find an analogous statement for PQR a general triangle.

    Some miscellaneous related problems:

37. First, some exercises using the methods of problem 12 (rotations, vectors). Given $\triangle ABC$, erect equilateral triangles $\triangle ABD$ and $\triangle BCE$ externally. Show that the midpoints of AC, BD and BE from an equilateral triangle.

38. Given $\triangle ABC$, erect equilateral triangles $\triangle ABD$ and $\triangle BCE$ externally and $\triangle AFC$ internally. What are the angles of the triangle formed by the center of $\triangle AFC$, D and E?

39. Given $\triangle ABC$, erect an equilateral triangle on side AB and a regular hexagon on side AC. Let their centers be D and E, and let the midpoint of BC be F. Find $\angle DFE$.

40. Here's an extension of problem 2. Let $\triangle ABC$ be equilateral and let P be a point on the arc BC of its circumcircle. Let AP intersect BC at Q. Show that $\frac{1}{PQ} = \frac{1}{BP} + \frac{1}{CP}$.

41. (Morley Triangle) This is a beautiful problem about equilateral triangles that's unrelated to the above, but nonetheless very beautiful. Given $\triangle ABC$, construct the internal and external angle trisectors. Let the internal angle trisectors of $\angle B$ and $\angle C$ closer to side BC intersect at A_1. Define B_1, C_1 similarly. Show that $\triangle A_1B_1C_1$ is equilateral. Let the external trisectors of $\angle B$ and $\angle C$ closer to side BC intersect at A_2. Define B_2, C_2 similarly. Show that $\triangle A_2B_2C_2$ is equilateral.

    Finally, let the external angle trisector of $\angle B$ closer to AB and the internal angle trisector of $\angle A$ closer to AB intersect at C_3, and the external angle trisector of $\angle C$ closer to AC and the internal angle trisector of $\angle A$ closer to AC intersect at B_3. Show $\triangle A_2B_3C_3$ is equilateral.

    These next two problems are reminiscent of problem 12.

42. Given a centrally symmetric hexagon, erect regular hexagons externally on the sides. Draw the segments connecting the centers of hexagons on adjacent sides. Show that the midpoints of these segments form a regular hexagon.

43. Let $ABCDEF$ be a hexagon inscribed in a circle of radius R with $AB = CD = EF = R$. Show that the midpoints of BC, DE, and FA form an equilateral triangle.

44. The third pedal triangle is similar to the original triangle.

45. Let A, B, C, P, Q, R be six points in the plane. If the circumcircles of ABR, AQC, PBC are concurrent, then the circumcircles of PQC, PBR, AQR are concurrent.

46. *Let the triple P, Q, R have Miquel point M with respect to triangle ABC. For any point X, let AX intersect the circumcircle of AQR at A', and define B', C' similarly. Then M, X, A', B', C' lie on a circle.