Problems about the Topology of Surfaces

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These vary somewhat in difficulty, but should all be fairly accessible.

Applications of Euler's Formula

1. Using Euler's Formula, prove that you can't connect eight points pairwise on a torus, in such a way that none of the connecting curves intersect. ("The complete graph K_8 cannot be embedded in a torus.")
2. With the same stipulation as the previous problem, connect five points pairwise on a torus; then six; then seven (hard!).
3. What's the maximum number of points that can be connected pairwise on the projective plane? On a Klein bottle?
4. Prove that any triangulation of the torus must have at least 7 vertices. That is, your solution to the previous problem (connecting 7 vertices pairwise on the torus) constitutes a *minimal triangulation* of the torus. What's the analogous result for the projective plane?

Other problems

5. If Σ_1 and Σ_2 are surfaces, prove that $\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$.
6. (Not easy)
   (a) By cutting and pasting polygons, show that $T \# P = P \# P \# P$.
   (b) Here's another approach to the same result. By cutting out a disk from $T \# P$ and $P \# P \# P$, we obtain the connected sum of a torus and a Möbius strip, and the connected sum of a Klein bottle and a Möbius strip, respectively. Show that these last two connected sums are homeomorphic. This implies (by gluing the disk back in) that $T \# P = P \# P \# P$.
7. (a) Comb a hairy torus without singularities. Then do the same for a hairy Klein bottle.
   (b) Comb a hairy projective plane in such a way that there's only one singularity.
   (c) Comb a hairy sphere in such a way that there's only one singularity.