USAMO 2010 PREPARATION SESSION
APRIL 6TH, 2010

Problem 1: Let ABC be a triangle such that $\angle A = 2\angle B$. Prove that $a^2 = b(b+c)$, where a, b, c are length of sides BC, AC, AB respectively.

Problem 2: Find all integers m, n such that

$$3 \cdot 2^m + 1 = n^2$$

Problem 3: For non-negative integers $a < b$ let $M(a,b)$ be the arithmetic mean of $\sqrt{i^2+3i+3}$ for $a \leq i \leq b$. Compute the whole part of $M(a,b)$ as a closed form function of a and b.

Problem 4: Find all functions $f : R^+ \Rightarrow R^+$ such that

$$(1 + yf(x))(1 - yf(x+y)) = 1$$

for all $x, y \in R^+$, where R^+ is a set of all positive real numbers.

Problem 5: In subset A of set $1, 2, \ldots, 2010$, the difference of any two elements is not a prime number. What is the maximum number of elements that can be in A.

Problem 6: Consider $n > 2$ checkers placed at the vertices of a regular n-gon. Each checker is colored red on one side and blue on another side. In one move, one can choose any three consecutive checkers and flip them up side down. A "situation" is any configuration of checkers (situations that are different only by a rotation, are considered different). How many situations can be reached from a given one with a finite number of moves?

Problem 7: Let a, b, and c be non-negative real numbers and x, y, and z be positive real numbers such that $a+b+c = x+y+z$. Prove that

$$\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \geq a+b+c$$

Problem 8: Let O be the center of inscribed circle of an acute triangle ABC. Let points A_0, B_0, C_0 be the centers of circumscribed circles around triangles BCO, ACO, ABO respectively. Prove that lines AA_0, BB_0, CC_0 intersect in one point.

Problem 9: Prove that for any natural number k, there exists a natural number n such that n has exactly k different prime factors and $2^{n^2}+1$ is divisible by n^3.

Problem 10: Let I be the center of circle w inscribed in trapezoid $ABCD$. Sides AD and BC intersect at point R. Let P and Q be the tangent points of W with sides AB and CD, respectively. Let the line passing through P and perpendicular to PR intersect lines AI and BI at points A_0 and B_0, respectively. Also, let the line passing through Q and perpendicular to QR intersect lines CI and DI at points C_0 and D_0, respectively. Prove that $A_0D_0 = B_0C_0$.