

Problem List: Topological Surfaces
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Some of these problems come from the book, The Shape of Space, by Jeff Weeks. You can find additional fascinating problems and ideas in that book.

How different are a torus and a sphere?

1. Can you connect each of three houses to each of three utilities with lines such a way that the lines never cross, **if the houses and utilities are on a torus**? The lines are allowed to wiggle and curve but must stay on the surface of the torus. You are not allowed to connect houses to houses or utilities to utilities.
2. The four color theorem states that four colors are enough to color any map on the sphere so that no two adjacent regions have the same color. (Regions are considered adjacent if they share a section of their border. Regions that just share a point are not considered adjacent.) How many colors are enough to color any map **on a torus** so that no two adjacent regions have the same color? (This is hard.)

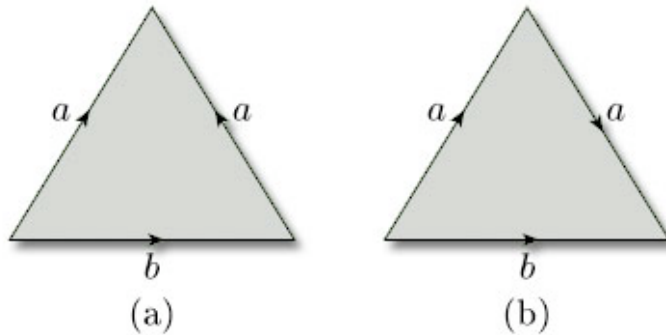
Tic Tac Toe on a Torus and on a Klein Bottle

3. A Cat's Game in Tic Tac Toe is a game where neither side wins, even though the board is filled up with X's and O's. Is it possible to have a Cat's Game in Tic Tac Toe on the torus?
4. How many essentially different opening moves does the first player have in Tic Tac Toe on the torus? How many different responses does the second player have?
5. Is there a winning strategy for the first player in Tic Tac Toe on the torus? That is, is it possible for the first player to win no matter what the second player does? Does it change your answer if the first player is required to start on the center square?
6. Play Tic Tac Toe on the Klein Bottle with a friend. Is it possible to get a Cat's Game? Is there a winning strategy for the first player?

Gluing Surfaces

7. What do you get when you cut a Mobius band around the center circle?
8. What do you get when you cut a Klein bottle in half? Does it depend on which way you cut it?
9. What do you get when you take the connected sum of two projective planes? (Hint: First, figure out what you get when you remove a disk from a projective plane. Then figure out what you get when you glue two of these together.)
10. (Hard) Show that the connected sum of a torus and a projective plane is the same as the connected sum of a Klein bottle and a projective plane.

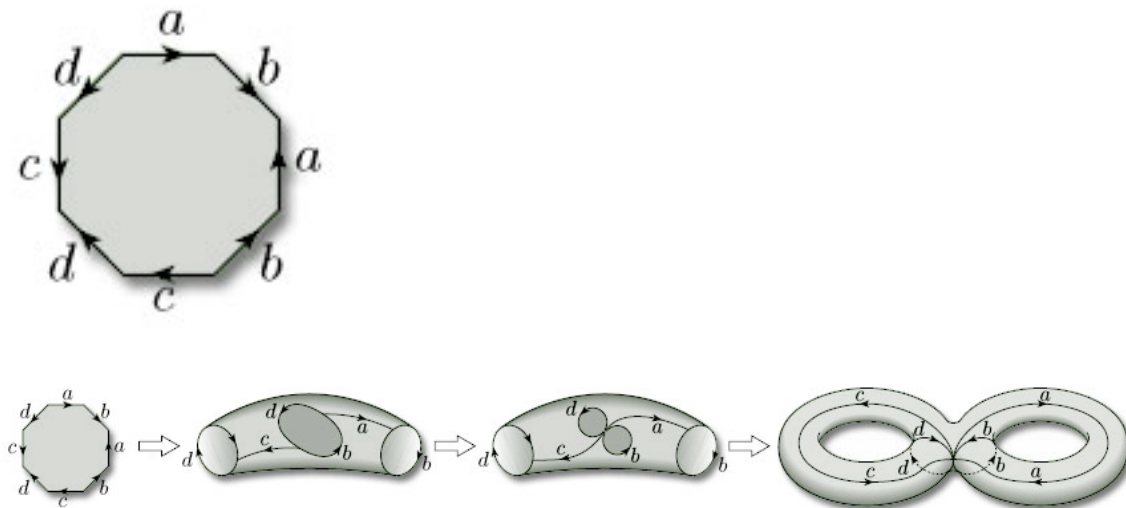
11. Which two surfaces are obtained by gluing the edges of each triangle as shown? You get two different surfaces, one for each triangle. Don't glue the two triangles together unless you really want to. Side b is not glued to anything.



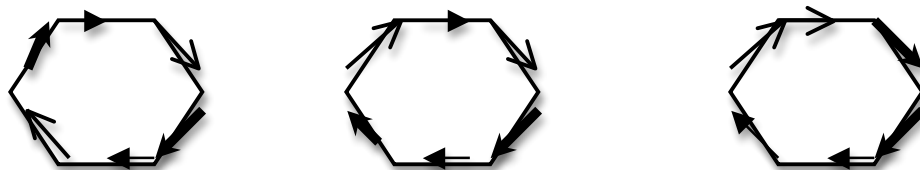
Hint: in (b), you might consider cutting the triangle before identifying the edges and then repairing the cut.

Name That Surface

12. An octahedron glued in the following pattern makes a 2-holed torus. Use the picture to convince yourself.



13. What surfaces do you get from each of the three hexagons with the following gluing patterns? On each hexagon, sides with matching arrows get glued together so that the arrow directions match. How can you tell from a gluing pattern if a surface is orientable?



Twisted Chess (from the Shape of Space)

14. Find a friend and play a few games of chess on a torus. The usual starting position doesn't work (try it and you'll see why), but you can use the starting position below or make up a good one of your own. All pieces move normally, except the pawns, which can move one space in a straight line in any direction, and captures by moving one space diagonally in any direction.

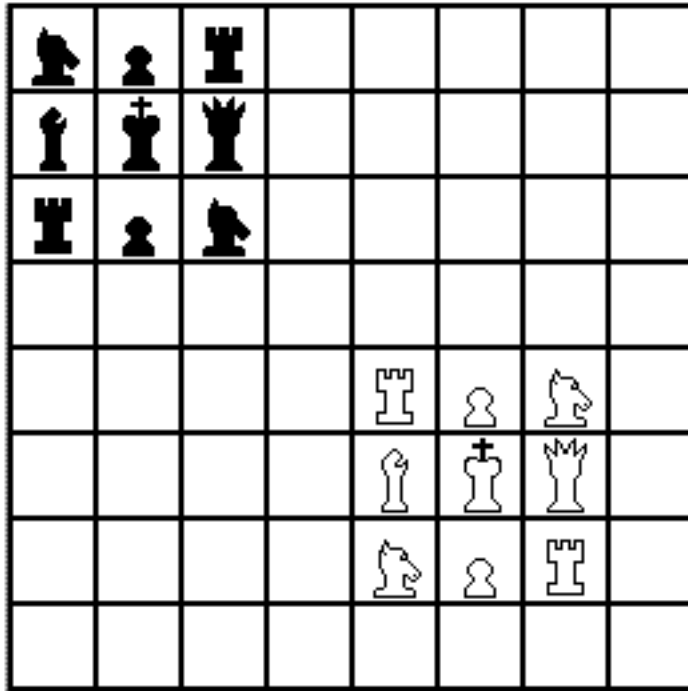


Figure 5

15. When playing torus chess, if a bishop goes out the upper right-hand corner of the chessboard, where does he return?
16. In torus chess, is it possible for a knight and a bishop to simultaneously threaten each other?
17. Try chess on a Klein bottle. Use the starting position above, and glue the left and right side as you would for a torus, but glue the top and bottom with a flip.
18. If a bishop goes out the upper right-hand corner in Klein bottle chess, where does he return?
19. In Klein bottle chess, starting from the initial position above, can the white bishop capture the black rook in one move? Starting from the initial position, can a black knight capture a white rook in one move?

Word Searches

20. Do a word search on a torus. Remember that the top and the bottom are glued together and the left and right sides are glued together. Find the following words:

bat	pig
camel	possum
cat	puma
jaguar	rat
llama	seal
ox	shrew
panda	tapir

h	l	m	e	a	i
n	a	l	n	b	r
j	l	d	a	e	a
c	a	t	w	m	t
x	e	g	i	p	a
o	s	s	u	m	p

21. Try this word search on the Klein bottle. The arrows show how the sides are glued together. (Hint: on a Klein bottle, letters and their mirror images are not distinguishable, so “d” and “b” are the same, and so are “p” and “q”.)

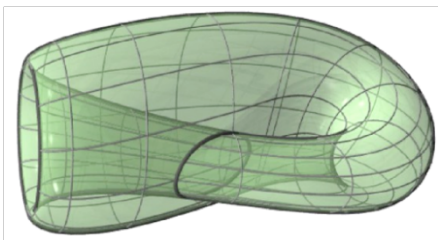
ash	lilac
birch	maple
cedar	oak
elm	palm
fir	pine
larch	poplar



h	p	m	a	c	m
a	d	e	c	k	r
a	d	n	f	e	l
a	l	i	l	m	c
s	r	q	r	e	a
h	a	h	o	c	l



These word searches were taken from Torus Games at <http://geometrygames.org/TorusGames/index.html> . Download Torus Games to play more.



A Klein bottle