# Mountain climbing with an enemy:
## Several problems with a rope and a couple of nails

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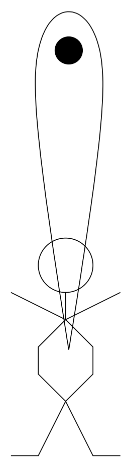
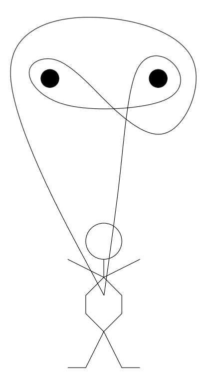


This figure represents a bad guy hanging on a rope wound around a nail hammered into a climbing wall. The good news is: if the nail falls out the bad guy will fall down.

**Problem.** Two nails are hammered into the climbing wall. Can you wind the rope around them in such a way that if one nail falls out the guy will fall down (whichever nail it is)?

**Solution:**

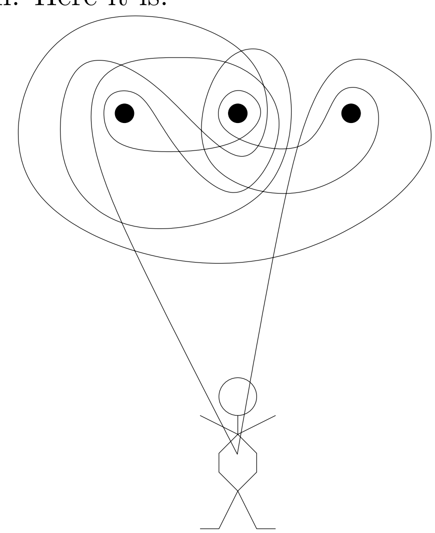


During the session this solution and the following ones are tested experimentally.

**Problem.** Three nails are hammered into the climbing wall. Can you wind the rope around them in such a way that if any two nails fall out the guy will fall down?

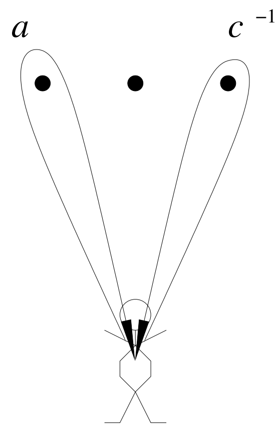
**Problem.** Three nails are hammered into the climbing wall. Can you wind the rope around them in such a way that if some nail (only one!) falls out the guy will fall down?

This is the most important problem of the session. But you know what, I'll give you a solution. Here it is:

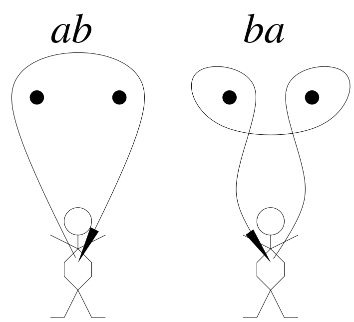


Is it the only possible solution? No it is not. But it is the simplest one. So now we are in an awkward situation: the problem asks you to construct an example of rope winding. I give just such an example. But you are not much wiser for that. Unless you have an exceptionally good mental representation of rope movements you will probably even be anable to check that my solution actually works.

So here is where *math* comes into play. Let's denote the three nails by $a$, $b$, and $c$. Further, denote a loop that goes clockwise (respectively counterclockwise) around the nail $a$ by $a$ (respectively $a^{-1}$). And similarly for the nails $b$ and $c$.




A more complicated loop will be denoted by a sequence of letters $a$, $a^{-1}$, $b$, $b^{-1}$, $c$, $c^{-1}$. For instance, the loop that goes clockwise first around the nail $a$ and then around the nail $b$ is denoted by $ab$. A loop that goes four times clockwise around the nail $c$ is denoted by $cccc$ or $c^4$. The "trivial" loop that does not go around any nails is denoted by 1. If a loop goes around, say, the nail $b$ first clockwise and then counterclockwise it is denoted by $bb^{-1}$. Note that this loop is actually trivial (a guy hanging on a loop like that will fall down immediately). Therefore we can write $aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = cc^{-1} = c^{-1}c = 1$. Our notation helps us remember this relation. Note, however, that $ab$ and $ba$ are two different loops:




**Problem.** Encode the loops from the previous problems and figures by sequences of letters.

**Answer.** For the first picture (with one nail) it's $a$. For the second picture (with two nails) it's $aba^{-1}b^{-1}$. For the third picture (with three nails) it's $aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$.

**Problem.** If we are given a word denoting some loop, how do we construct a word that denotes the same loop but with reversed direction?

Now consider the loop $aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$ that was the solution to the last 3-nail problem. What will happen to this loop if the nail $a$ falls out of the wall? Well, we don't care any longer if the loop goes around the hole named $a$. So we can just erase the letters $a$ and $a^{-1}$ from the word. The remaining word $bb^{-1}cbb^{-1}c^{-1}$ describes how the loop goes around the remaining nails $b$ and $c$. This word can be simplified, because we know that $bb^{-1} = 1$. So this word is equivalent to $cc^{-1}$. But this is also equal to 1. So we have proved that the loop is actually trivial: that is, the bad guy will fall down as required.

**Exercise.** Check that if the nail $b$ or the nail $c$ falls out of the wall we also obtain trivial loops.

**Problem.** Four nails are hammered into the climbing wall. Wind the rope around them in such a way that if some nail falls out the guy will fall down.

To solve this problem the following definition may be helpful: the *commutator* of two loops $l_1$ and $l_2$ is the loop $l_1 l_2 l_1^{-1} l_2^{-1}$ obtained by the concatenation of the loop $l_1$, the loop $l_2$, the loop $l_1$ with reversed direction, and the loop $l_2$ with reversed direction. It is denoted by $[l_1, l_2]$.

**Problem.** Several nails are hammered into the climbing wall. Prove that one can wind the rope around them in such a way that if some nail falls out the guy will fall down.

**Problem.** How many letters does the *shortest* word that solves the $n$-nail problem have?

Note: I don't know the answer.

**Questions to think over.** Why can any loop be decomposed into a sequence of loops going around individual nails (to be encoded by letters)? Suppose a word that encodes a loop $l$ does not contain any simplifiable expression like $aa^{-1}$ or $c^{-1}c$. Why does it imply that the loop is nontrivial? These are two facts that we implicitely used above.

**P.S.** While we are on the theme of loops, let me tell you about a complicated space where there exists a nontrivial loop $l$ such that $l^2$ (a concatenation of two $l$'s) is trivial.

It is the space of all rotations of space about the origin. If we fix an orthonormal basis centered at the origin (or *frame* for shortness), its image under every rotation it another frame. Thus the space of rotations can be identified with the space of frames.

Let us try to understand this space. In space fix a vector of length $\alpha \leq 2\pi$. This vector determines a unique rotation by the angle $\alpha$ about the axis of the vector in the clockwise direction (if we are looking in the direction in which the vector is pointing). If the angle $\alpha$ equals 0, we get the "trivial" rotation (that does not rotate anything). The axis of rotation is then immaterial. This trivial rotation is encoded by the zero vector. (By a lucky coincidence, when the length of a vector equals 0 its direction is immaterial.) So far it seems that our space of rotations is identified with a ball or radius $2\pi$. However, this is not the end of the story.

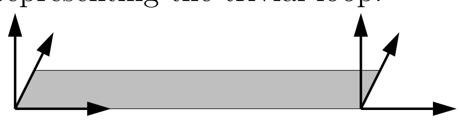
Indeed, if the angle $\alpha$ equals $2\pi$ then we also get the trivial rotation. Therefore we have to identify all the points on the boundary of the ball. What happens can be understood by analogy with a disc: if you glue together all the points on the boundary of a disc you get a sphere. In the case of a ball you get a 3-dimensional sphere.

Finally, a clockwise rotation by $\alpha$ is the same as a counterclockwise rotation by $2\pi - \alpha$. Therefore we have to identify pairwise the opposite points of the 3-dimensional sphere. This is hard to imagine, so let's think of the space of rotations as a 3-dimensional sphere, but bear in mind that two opposite points of the sphere represent the same rotation.

A loop in the space of rotations starts at the trivial rotation, travels on the sphere, and comes back to the same rotation. That is, either to the same point of the sphere or to the opposite point. If the loop comes back to the same point of the sphere, then it is a trivial loop: it can be moved continuously along the sphere until it is entirely condensed at the initial point. On the other hand, if the loop ends at the opposite point of the sphere, then it is nontrivial. A concatenation of two nontrivial loops brings us from the initial point to the opposite point of the sphere and then back again to the initial point. So the concatenation of two nontrivial loops is trivial.

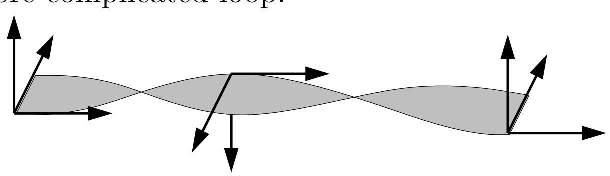
Can we test this fact experimentally, as we did with the rope and the nails? Yes we can. To do that we need to find a way to physically represent a loop in the space of rotations. Our physical model will be a ribbon. Every point of a ribbon encodes a frame: the first vector points along the ribbon, the second one across the ribbon, and the third one perpendicular to the ribbon. As we move along the ribbon this frame changes continuously, in other words, we get a loop in the space of frames. (Recall that, by definition, all our frames are supposed to be centered at the origin. This means that all the frames encoded by a ribbon should be mentally transported to the origin. So the position of the ribbon in space does not matter, it's only the way it twists that is important.)

Here is a ribbon representing the trivial loop:



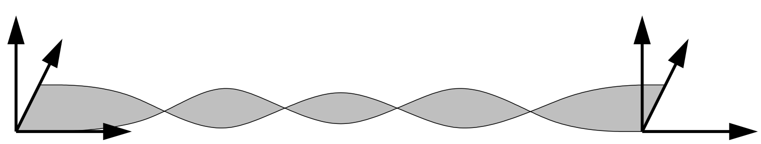
All the frames along the ribbon are the same.

Here is a more complicated loop:



To see if it is trivial or not, we can play the following game: you can move the ribbon freely, but without *turning* its ends (because the loop has a fixed beginning and end in the space of frames). Can you straighten the ribbon? Try it. You'll see that it is impossible.

Now here is a ribbon that represents a concatenation of two nontrivial loops:



Play the same game with this ribbon and you'll see that you can straighten it.