MEDIANS SURRENDER AT THE OLYMPICS

Geometry at the Bay Area Mathematical Olympiad[1]

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Abstract: Have you heard of the expression "*center of mass of* $\triangle ABC$"? Very likely you have! If M is the midpoint of side BC, then segment AM is called a *median* of the triangle. If you draw the three medians very carefully, you will discover that they meet at some point G. If you hang your triangle (conveniently made of cardboard!) on a string from this point G, you will discover that the triangle stays horizontal to the floor! This is why point G is called the *centroid* (or *center of mass*) of $\triangle ABC$. Try it! In this session we will see how the three medians and the centroid challenge students in puzzling geometry problems from the *Bay Area Mathematical Olympiad (BAMO)*, only to surrender to students' creative solutions via geometric transformation, extra constructions, and dust-covered century-old theorems!

Medians and Centroids

(1) **(BAMO '00)** Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F; let k_2 be the circle passing through points B, E, and D; and let k_3 be the circle passing through points C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

(2) **(BAMO '05)** If two medians in a triangle are equal in length, prove that the triangle is isosceles.

(3) **(BAMO '06)** In $\triangle ABC$, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.

Geometry on the Circle

(4) **(BAMO '99)** Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0, a)$ and $B = (0, b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection on the line through P and A with the x-axis, and let $O = (0, 0)$. Prove that $\angle BQP = \angle BOP$.

(5) **(BAMO '99, shortlisted IMO '98)** Let $ABCD$ be a cyclic quadrilateral (i.e., it can be inscribed in a circle). Let E and F be variable points o the sides AB and CD, respectively, such that $AE/EB = CF/FD$. Let P be the point on segment EF such that $PE/PF = AB/CD$. Prove that the ratio between the areas of $\triangle APD$ and $\triangle BPC$ does not depend on the choice of E and F.

(6) **(BAMO '02)** Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to $\triangle ABC$. If M is the center of the square, find the measure of $\angle MBC$.

Projective Geometry?

(7) **(BAMO '01)** Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y and Z are collinear (i.e., lie on the same line).

(8) **(BAMO '06)** In $\triangle ABC$, choose point A_1 on side BC, point B_1 on side CA, and point C_1 on side AB in such a way that the three segments AA_1, BB_1, and CC_1 intersect in one point P. Prove that P is the centroid of $\triangle ABC$ if and only if P is the centroid of $\triangle A_1B_1C_1$.

[1]At the Berkeley Math Circle, April 27 2010.