Generating series and nice number sequences

Definition. A *formal power series* is an expression of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots,$$

where $a_0, a_1, a_2, \dots$ are numbers.

Notation. We denote by e^x the power series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

1. Define the sum and the product of two formal power series.

2. Prove that $e^x \cdot e^x = e^{2x}$, $e^x \cdot e^{-x} = 1$.

3. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots,$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

be two power series. Prove that if $b_0 \neq 0$, there exists a unique power series $h(x)$ such that $g(x)h(x) = f(x)$. This series is denoted by $f(x)/g(x)$.

4. Compute $1/(1-x)$, $1/(1-x)^2$, $1/(1-ax)$.

5. The sequence of *Fibonacci numbers* is defined by $a_0 = a_1 = 1$, $a_{n+2} = a_n + a_{n+1}$. Its first terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

Let $f(x) = a_0 + a_1 x + \dots$ be the *generating series* for the Fibonacci numbers. Compute $f(x) \cdot (1 - x - x^2)$. Prove that

$$f(x) = \frac{1}{\sqrt{5}} \left[\frac{\frac{1+\sqrt{5}}{2}}{1 - \frac{1+\sqrt{5}}{2}x} - \frac{\frac{1-\sqrt{5}}{2}}{1 - \frac{1-\sqrt{5}}{2}x} \right].$$

Find an explicit formula for a_n. Prove that a_{n+1}/a_n tends to $(1+\sqrt{5})/2$ (the *golden ratio*) as n tends to ∞.

Definition. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ be a formal power series. Its *derivative* is the formal power series

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots.$$

6. Compute the derivatives of e^x, $1/(1-x)$.

7. Prove that $(fg)' = f'g + fg'$ for any power series f, g.

8. If $f(x) = a_0 + a_1x + \ldots$ what is the free term of its n-th derivative?

9. Let

$$f(x) = 1 + a_1x + a_2x^2 + \ldots$$

be a formal power series with free term equal to 1. Prove that there exists a unique power series $g(x)$ such that $f(x) = g(x)^2$. This power series is denoted by $\sqrt{f(x)}$.

10. Prove that

$$(\sqrt{1-x})' = -\frac{1}{2}\frac{1}{\sqrt{1-x}}.$$

Compute the derivatives of $\left(\sqrt{1-x}\right)^n$ and $1/\left(\sqrt{1-x}\right)^n$.

11. Prove that

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{2^n\, n!} x^n.$$

Definition. The nth *Catalan number* C_n is the number of all possible sequences composed of $2n$ correctly matched parentheses. The first Catalan numbers are $C_0 = 1$ (by convention); $C_1 = 1$:

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$C_2 = 2$:

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$C_3 = 5$:

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and $C_4 = 14$:

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12. Prove the equality

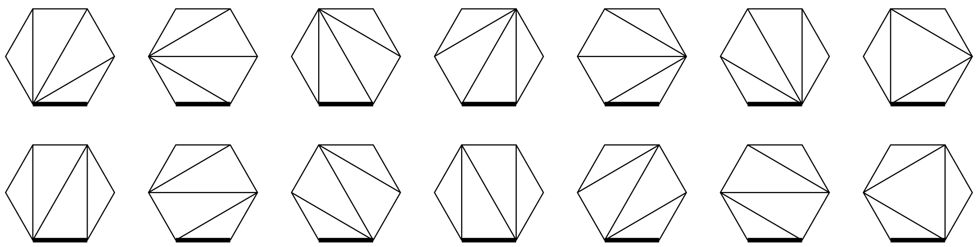
$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots C_{n-1} C_0.$$

13. Let $f(x) = C_0 + C_1 x + \dots$ be the generating series for the Catalan numbers. Compute $xf^2 - f$. Prove that

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Find a closed formula for C_n.

14. Prove that the number of triangulations of an $(n+2)$-gon with a distinguished side equals C_n. For instance, here are the 14 triangulations of a hexagon:



15. Prove that

$$\begin{aligned}
1^0 + 2^0 + 3^0 + \dots + (n-1)^0 &= n - 1, \\
1^1 + 2^1 + 3^1 + \dots + (n-1)^1 &= \frac{1}{2}n^2 - \frac{1}{2}n, \\
1^2 + 2^2 + 3^2 + \dots + (n-1)^2 &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n, \\
1^3 + 2^3 + 3^3 + \dots + (n-1)^3 &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2.
\end{aligned}$$

16. Prove that for any $k \geq 0$ there exists a polynomial $P_k(n)$ of degree $k+1$ such that

$$1^k + \ldots + (n-1)^k = P_k(n).$$

Definition. The coefficient of n in P_k is called the nth *Bernoulli number* and denoted by B_n. The first Bernoulli numbers are

$$1,\ -\frac{1}{2},\ \frac{1}{6},\ 0,\ \frac{-1}{30},\ 0,\ \frac{1}{42},\ 0,\ \frac{-1}{30},\ 0,\ \frac{5}{66},\ 0,\ \frac{-691}{2730},\ 0,\ \frac{7}{6},\ 0,\ \frac{-3617}{510},\ 0,\ \frac{43867}{798},\ldots$$

17. Prove the recursion relation for $k \geq 2$:

$$\binom{k}{0}B_0 + \binom{k}{1}B_1 + \ldots + \binom{k}{k-1}B_{k-1} = 0.$$

18. Prove that

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

19. Prove that the power series

$$f(x) = \frac{x}{e^x - 1} + \frac{x}{2}$$

is even, that is, $f(-x) = f(x)$. Deduce that all odd Bernoulli numbers vanish, except B_1.

20. Find all the coefficients of the polynomials P_k.