The Dynamics of Continued Fractions

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February 16, 2010

Notation. A (simple) *continued fraction* is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_0, a_1, \ldots$ are positive integers. We let

$$\frac{p_1}{q_1} = a_0, \frac{p_2}{q_2} = a_0 + \frac{1}{a_1}, \frac{p_3}{q_3} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}, \ldots$$

be the reduced forms of the *convergents* of the continued fraction, and

$$x_0 = x, x_1 = \frac{1}{x - a_0}, x_2 = \cfrac{1}{\cfrac{1}{x - a_0} - a_1}, \ldots$$

be the *remainders* of the continued fraction.

Problems

1. Prove that

$$p_{n+1} = a_n p_n + p_{n-1}$$
$$q_{n+1} = a_n q_n + q_{n-1}$$

for $n \geq 2$. If these equations are to hold also for $n = 0$ and $n = 1$, what must p_0/q_0 and p_{-1}/q_{-1} be?

2. Prove that

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}}$$

for $n \geq 1$.

3. Prove that if $x = \sqrt{k}$ for some positive integer k, not a perfect square, then all the remainders x_i ($i \geq 1$) have the form

$$\frac{\sqrt{k} + P}{Q}$$

for positive integers P and Q such that $Q \mid (k - P^2)$.

For problems 4–6, k is a non-square positive integer, $d = \lfloor \sqrt{k} \rfloor$, and $f(x) = \frac{dx+k}{x+d}$.

4. Prove that if $a > 0$ is any real number, then the sequence

$$\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

converges to the limit $\sqrt{k}$.

5. Prove that $(k - d^2) \mid 4d^2$ if and only if

$$k = \frac{s^2 v(vm^2 + 4)}{4}$$

for integers s, v, m satisfying $\gcd(m, s) = 1$, $m > s$, and $2 \mid svm$.

6. Prove that $(k - d^2) \mid 2d$ if and only if the remainders x_1 and x_3 in the continued fraction expansion of $x = \sqrt{k}$ are equal. (This necessarily implies that the continued fraction $[a_0, a_1, a_2, \cdots]$ has period 2 after the first term.)