Games and Their Values

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In all of these games, there are two players who take turns, and if it's your turn and you're unable to move, you lose. The question "Who wins?" means: does the player going first always win? Going second? The "blue" player? The "red" player?

Blue-Red Hackenbush

In this game, you draw the ground, and a collection of blue and red edges. The players take turns erasing an edge of their color, and also erasing any edge which has become disconnected from the ground.

1. Who wins if there are no edges? We call this game "0", for the obvious reason.

2. Who wins if there is one blue edge? We call this game "1".

3. What game should be called –1?

4. What game should be called 2? Can you think of several different possibilities?

5. What is the value, compared with these games, of BR? (That represents a stack with B connected to the ground and R connected to the B).

6. Can you figure out the value of
   R R
   B B R
   (which represents two stacks with R on top of B, and one with B on the ground)? If so, this might help you answer the previous problem.

7. Can you make a game with value 1/4? 5/8? 11/16? 1/3?

Notational Interlude: {1|2} means that blue's best move is to a game with value 1, and red's best move is to a game with value 2.

Blue-Red-Green Hackenbush

Green edges may be taken by either player.

8. What is the value of G?

9. What is the value of BG?

10. What is the value of GB?

11. What is the value of G + GB?

12. What is the value of G + GB + GB?

Breaking Chocolate

There's a rectangle of chocolate squares, with r rows and c columns. One player may break one rectangle along any vertical line (so initially, he has c-1 choices of where to break it). The other player may break one rectangle along any horizontal line (so initially, she has r-1 choices of where to break it). After the rectangle is broken, at each turn only one of the smaller rectangles may be broken. The loser is the first one who cannot move (or, alternatively, the winner is the last one to break the chocolate).

13. Who wins (1,1)? Yes, this is the trivial 0 game again.

14. Who wins (2,2)? (3,3)? (4,4)? (5,5)? What are the values of those games?

15. How about (2,3)? (3,2)?

16. Can you solve this game in general for (m, n)?

Taking Candy

17. Two players take turns removing one from a pile of N candies. Whoever takes the last candy wins. Who should win this game, given a value of N?

18. Two players take turns removing one, two, or three candies from a pile of N. Now what is the winning strategy? (Again, getting the last candy wins).

19. Two players take turns removing 5, 7, or 11 candies from a pile of N. Now what is the winning strategy? (Here winning means that there are fewer than 5 candies left, so neither player can move).

20. Two players take turns removing any number of candies less than (not equal to) half of the total number on the table; or they can take one in any case. Who should get the last candy now?

A Miscellany

21. The divisor game: To play game N, both players agree in advance on a number, N. Then each player in turn names a divisor of N, without naming a multiple of any previously named number. The player who names 1 ends the game and thus loses. Who should win for each value of N?

22. Chairs: two players take turns seating people, but the people smell bad so nobody will sit next to an already-seated person. Whoever seats the last person wins.

23. Split a pile: A move consists of splitting one pile into two, neither of which are equal in size to any other piles. The player who splits the last pile wins.