Classical inequalities.

Arithmetic mean-geometric mean [AM-GM] inequality. For any nonnegative numbers a_1, …, a_n,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Power mean inequality. For positive numbers a_1, …, a_n, and a real number α, let

$$M_\alpha(a_1, \ldots, a_n) := \begin{cases} \left(\frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha} & \alpha \neq 0 \\ \sqrt[n]{a_1 \cdots a_n}, & \alpha = 0. \end{cases}$$

Then M_α is an increasing function of α unless $a_1 = \cdots = a_n$, in which case M_α is constant.

Cauchy's inequality. For arbitrary real numbers a_1, …, a_n, b_1, …, b_n,

$$(a_1 b_1 + \cdots a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

Furthermore, equality holds if and only if the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

Cauchy's inequality is equivalent to the triangle inequality for the 2-norm.

Triangle inequality. For any two vectors x, y in $\mathbb{R}^n$,

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

Definition. Suppose that f is a real-valued function defined on an interval and, for any points x, y in the interval,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}.$$

Then f is *convex.*

Convex functions have useful properties.

Jensen's inequality. If w_1, …, w_n are positive numbers satisfying $w_1 + \cdots + w_n = 1$, and x_1, …, x_n are any n points in an interval where f is convex, then

$$f(w_1 x_1 + \cdots + w_n x_n) \leq w_1 f(x_1) + \cdots + w_n f(x_n).$$

Points of maximum. If f is convex on $[a, b]$, then the maximum value of f is taken at one of the endpoints, i.e.,

$$f(x) \leq \max\{f(a), f(b)\}.$$

Weighted AM-GM inequality. If x_1, …, x_n are nonnegative real numbers and w_1, …, w_n are positive numbers satisfying $w_1 + \cdots + w_n = 1$, then

$$\prod_{i=1}^{n} x_i^{w_i} \leq \sum_{i=1}^{n} w_i x_i.$$

Equality holds if and only if $x_1 = \cdots = x_n$.

Theorem. If a and b are nonnegative numbers and p, $q > 1$ satisfy $1/p + 1/q = 1$, then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab, \tag{1}$$

with equality if and only if $a^p = b^q$.

Hölder's inequality. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be nonnegative and let p, $q > 1$ satisfy $1/p + 1/q = 1$. Then

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q}.$$

Minkowski's inequality. If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are nonnegative numbers and $p \geq 1$, then

$$\left(\sum_{i=1}^n (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} + \left(\sum_{i=1}^n y_i^p\right)^{1/p}.$$

Theorem [Hölder]. Let $X = (x_{ij})$ be an $m \times n$ matrix with nonnegative elements and let $w_1, \ldots, w_n$ be positive numbers satisfying $w_1 + \cdots + w_n = 1$. Then

$$\sum_{i=1}^m \prod_{j=1}^n x_{ij}^{w_j} \leq \prod_{j=1}^n \left(\sum_{i=1}^m x_{ij}\right)^{w_j}.$$

Rearrangement inequality. Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be two sequences of real numbers and suppose $a_1 \leq a_2 \leq \cdots \leq a_n$. For each permutation π of $\{1, 2, \ldots, n\}$ let

$$\Sigma(\pi) := \sum_{k=1}^n a_k b_{\pi(k)}.$$

Then Σ is largest when $b_{\pi(1)} \leq \cdots \leq b_{\pi(n)}$ and smallest when $b_{\pi(1)} \geq \cdots \geq b_{\pi(n)}$.

Examples.

1. Let $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that

   $$n(n+1)^{1/n} < n + H_n$$

   for every $n \in \mathbb{N}$. Hint: AM-GM inequality.

2. Show that if $0 \leq a, b, c \leq 1$, then

   $$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (a-1)(b-1)(c-1) \leq 1.$$

   Hint: convexity.

3. Let $n \in \mathbb{Z}$, $n \neq 0, -1$. Prove that if

   $$\frac{\sin^{2n+2} A}{\sin^{2n} B} + \frac{\cos^{2n+2} A}{\cos^{2n} B} = 1$$

   holds, then it holds for all $n \in \mathbb{Z}$. Hint: inequality (1).

4. Let $a := (m^{m+1} + n^{n+1})/(m^m + n^n)$ where $m,\ n \in \mathbb{N}$. Prove that $a^m + a^n \geq m^m + n^n$. Hint: Bernoulli's inequality.

5. For $x,\ y,\ z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds. Hint: find and use symmetry.

6. For a positive number x and an integer n, prove that

$$\sum_{k=1}^{n} \frac{\lfloor kx \rfloor}{k} \leq \lfloor nx \rfloor.$$

Hint: rearrangement inequality.