Note: All objects in this handout are planar - i.e. they lie in the usual plane. We say that several points are collinear if they lie on a line. Similarly, several points are concyclic if they lie on a circle; an inscribed (cyclic) polygon has its vertices lying on a circle. If three distinct points \( A, B \) and \( C \) are collinear, then the directed ratio \( \frac{AB}{CB} \) is the ratio of the lengths of segments \( AB \) and \( CB \), taken with a sign “+” if the segments have the same direction (i.e. \( B \) is not between \( A \) and \( C \)), and with a sign “−” if the segments have opposite directions (i.e. \( B \) is between \( A \) and \( C \)). Several objects (lines, circles, etc.) are concurrent if they all intersect in some point.

1. (Menelaus) Let \( A_1, B_1 \) and \( C_1 \) be three points on the sides \( BC, CA \) and \( AB \) of \( \triangle ABC \). Prove that they are collinear (cf. Fig. 1) iff

\[
\frac{AB_1}{CB_1} \cdot \frac{CA_1}{BA_1} \cdot \frac{BC_1}{AC_1} = 1.
\]

2. (a) Prove that the interior angle bisectors of two angles of nonisosceles \( \triangle ABC \) and the exterior angle bisector of the third angle intersect the opposite sides (or their continuations) of \( \triangle ABC \) in three collinear points. (cf. Fig. 2a)

(b) Prove that the exterior angle bisectors of nonisosceles \( \triangle ABC \) intersect the continuations of opposite sides of \( \triangle ABC \) in three collinear points.

(c) Prove that the tangents at the vertices of nonequilatera \( \triangle ABC \) to the circum-circle of \( \triangle ABC \) intersect the continuations of opposite sides of \( \triangle ABC \) in three collinear points. (cf. Fig. 2c)

3. (Pascal) If the hexagon \( ABCDEF \) is cyclic and its opposite sides, \( AB \) and \( DE \), \( BC \) and \( EF \), \( CD \) and \( FA \), are pairwise not parallel, prove that their three points of intersection, \( X \), \( Y \) and \( Z \), are collinear. (cf. Fig. 3)

The same statement is true if the circle is replaced by an ellipse, hyperbola or parabola.\(^1\) The statement is also true if some of the vertices of the hexagon coincide

\(^1\)These all are conics, i.e. projectively equivalent to a circle.
– then replace the corresponding side of the hexagon by the tangent to the circle at the corresponding vertex. Thus, obtain the following:

(a) If $A = B$, $C = D$, $D = F$, deduce to Problem 2c.

(b) If $E = F$, formulate the property of any inscribed pentagon.

(c) If $A = F$ and $D = E$, for the inscribed quadrilateral $ABCD$ we have: the intersection points of $AB$ and the tangent at $D$, of $CD$ and the tangent at $A$, and of $BC$ and $AD$, are collinear.

(d) If $A = F$ and $C = D$, the intersection points of the pairs of opposite sides of an inscribed quadrilateral and the intersection of the tangents at two opposite vertices are collinear. (Actually, the tangents at any pair of opposite vertices should also work.)

4. (Desargues) $\triangle ABC$ and $\triangle A_1B_1C_1$ are positioned in such a way that lines $AA_1$, $BB_1$, and $CC_1$ intersect in a point $O$. If lines $AB$ and $A_1B_1$, $AC$ and $A_1C_1$, $BC$ and $B_1C_1$ are pairwise not parallel, prove that their points of intersection, $L$, $M$ and $N$, are collinear. (cf. Fig. 4)

Note: The incircle of $\triangle ABC$ is the circle inscribed in $\triangle ABC$ (i.e. tangent to all three sides of the triangle.) Its center is called the incenter of $\triangle ABC$; it lies on the angle bisectors of $\triangle ABC$. The excircle of $\triangle ABC$ tangent to side $AB$ is the circle tangent to side $AB$ and to the extensions of sides $BC$ and $AC$. Its center is called an excenter of $\triangle ABC$. On what bisectors does this excenter lie? The circumcircle of $\triangle ABC$ is the circle passing through the vertices $A$, $B$ and $C$. Its center is called the circumcenter of $\triangle ABC$; it lies on the perpendicular bisectors of the sides of the triangle.
5. Prove that the midpoint $K$ of the altitude $CH$ in $\triangle ABC$, the incenter $I$ of $\triangle ABC$, and the tangency point $T$ on $AB$ of the excircle of $\triangle ABC$ (tangent to side $AB$) are collinear. (cf. Fig. 5)

6. (Gauss’s line with respect to $l$) Line $l$ intersects the sides (or continuations of) $BC$, $CA$ and $AB$ of $\triangle ABC$ in points $P_1$, $P_2$ and $P_3$. Prove that the midpoints $M_1$, $M_2$ and $M_3$ of $AP_1$, $BP_2$ and $CP_3$ are collinear. (cf. Fig. 6)

7. Let $ABCD$ be a quadrilateral with perpendicular diagonals intersecting in $P$. The feet of the perpendiculars from $P$ to sides $AB$, $BC$, $CD$ and $DA$ are $P_1$, $P_2$, $P_3$ and $P_4$. Prove that lines $P_1P_2$, $P_3P_4$ and $CA$ are concurrent. (cf. Fig. 7)

8. (Simpson) Prove that the feet of the perpendiculars dropped from a point $M$ on the circumcircle $k$ of $\triangle ABC$ to the sides of the triangle are collinear. More generally, let $S$ be the area of $\triangle ABC$, $R$ – the circumradius, and $d$ – the radius of a circle $\epsilon$ concentric to $k$. Let $A_1$, $B_1$ and $C_1$ be the feet of the perpendiculars dropped from an arbitrary point on $\epsilon$ to the sides of $\triangle ABC$. Prove that the area $S_1$ of $\triangle A_1B_1C_1$ is given by the formula $S_1 = \frac{1}{4}S\left[1 - \frac{d^2}{R^2}\right]$. In particular, when $\epsilon = k$, then $S_1 = 0$, and hence $A_1$, $B_1$ and $C_1$ are collinear. (cf. Fig. 8)

9. (Salmon) Through a point $M$ on a circle $\epsilon$ draw three arbitrary chords $MA$, $MB$ and $MC$, and using each chord as a diameter, draw three new circles $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$. Prove that the pairwise intersections of the $\epsilon_i$’s (other than $M$) are collinear.

Note: Let $H$ be the orthocenter of $\triangle ABC$ (i.e. the intersection of the altitudes of the triangle.) The Euler circle of 9 points for $\triangle ABC$ is the circle passing through the midpoints of the sides of $\triangle ABC$, the midpoints of $AH$, $BH$ and $CH$, and the feet of the altitudes of $\triangle ABC$. In fact, the
center of this circle is the midpoint of $HO$ ($O$ is the circumcenter of $\triangle ABC$), and its radius is half of the circumradius of $\triangle ABC$. Why? (cf. Fig. 10a)

10. Prove that Simpson’s line of $\triangle ABC$ with respect to point $M$ on the circumscribed circle $k$ of $\triangle ABC$, line $MH$ where $H$ is the orthocenter of $\triangle ABC$, and the Euler circle of 9 points for $\triangle ABC$ are concurrent. (cf. Fig. 10b)

11. (Ceva) Let $A'$, $B'$ and $C'$ be three points on the sides (or continuations of) $BC$, $CA$, $AB$ of $\triangle ABC$. Prove that $AA'$, $BB'$, $CC'$ are concurrent or are parallel iff

$$\frac{AB'}{CB'} \cdot \frac{CA'}{BA'} \cdot \frac{BC'}{AC'} = -1.$$

12. (Gergonne’s point) Prove that the lines connecting the vertices of a triangle with the points of tangency of the inscribed circle are concurrent. (cf. Fig. 12)

13. (Nagel’s point) Prove that the lines connecting the vertices of a triangle with the corresponding points of tangency of the three externally inscribed circles are concurrent (cf. Fig. 13.) Note that these are also the three lines through the vertices of the triangle and dividing each its perimeter into two equal parts.

14. Let $M$ be an arbitrary point on side $AB$ of $\triangle ABC$. Let $P$ and $Q$ be the intersection points of the angle bisectors of $\angle BMC$ and $\angle AMC$ with sides $BC$ and $AC$, respectively. Prove that lines $AP$, $BQ$ and $CM$ are concurrent.
15. Let $A_1, B_1, C_1$ be points on the sides of an acute-angled $\triangle ABC$ so that the lines $AA_1, BB_1$ and $CC_1$ are concurrent. Prove that $CC_1$ is an altitude in $\triangle ABC$ iff it is the angle bisector of $\angle B_1C_1A_1$.

16. In the acute-angled $\triangle ABC$ a semicircle $k$ with center $O$ on side $AB$ is inscribed. Let $M$ and $N$ be the points of tangency of $k$ with sides $BC$ and $AC$. Prove that lines $AM$, $BN$ and the altitude $CD$ of $\triangle ABC$ are concurrent. (cf. Fig. 16)

17. A circle $k$ intersects side $AB$ of $\triangle ABC$ in $C_1$ and $C_2$, side $CA$ in $A_1$ and $A_2$, side $BC$ in $B_1$ and $B_2$. The order of these points on $k$ is: $A_1, A_2, B_1, B_2, C_2, C_1$. Prove that lines $AA_1$, $BB_1$, $CC_1$ are concurrent iff $AA_2$, $BB_2$, $CC_2$ are concurrent. (cf. Fig. 17)

18. Let the points of tangency of the incircle of $\triangle ABC$ with the sides $AB$, $BC$ and $CA$ be $C_1$, $A_1$ and $B_1$, and let $A_2$, $B_2$ and $C_2$ be their reflections across the incenter $I$ of $\triangle ABC$. Prove that lines $AA_2$, $BB_2$ and $CC_2$ are concurrent. (cf. Fig. 18)

19. (Gauss) If the two pairs of opposite sides of a quadrilateral $ABCD$ intersect in $E$ and $F$, prove that the midpoint $N$ of $EF$ lies on the line through the midpoints $L$ and $M$ of the diagonals $AC$ and $BD$. (cf. Fig. 19)

20. Point $P$ lies inside $\triangle ABC$. Lines $AP$, $BP$, $CP$ intersect the sides $BC$, $CA$, $AB$ in $A_1, B_1, C_1$, respectively, and $L, M, N, L_1, M_1, N_1$ are the midpoints of the segments $BC, CA, AB, B_1C_1, C_1A_1, A_1B_1$. Prove that $LL_1, MM_1$ and $NN_1$ are concurrent. (cf. Fig. 20)
21. Let $P$, $Q$ and $R$ be points on the sides $BC$, $CA$ and $AB$ of $\triangle ABC$. Let $O_1$, $O_2$ and $O_3$ be the circumcenters of $\triangle AQR$, $\triangle BRP$ and $\triangle CPQ$. Prove that $\triangle O_1O_2O_3 \sim \triangle ABC$. (cf. Fig. 21)

22. (IMO’81) Three congruent circles pass through point $P$ inside $\triangle ABC$. Each circle is inside $\triangle ABC$ and is tangent to two of its sides. Prove that the circumcenter $O$ and incenter $I$ of $\triangle ABC$ and $P$ are collinear. (cf. Fig. 22)

23. (Brianchon) If the hexagon $ABCDEF$ is circumscribed around a circle, prove that its three diagonals $AD, BE$ and $CF$ are concurrent. (cf. Fig. 23)

24. (Saint Petersburg Olympiad) Point $I$ is the incenter of $\triangle ABC$. Some circle with center $I$ intersects side $BC$ in $A_1$ and $A_2$, side $CA$ in $B_1$ and $B_2$, and side $AB$ in $C_1$ and $C_2$. The six points obtained in this way lie on the circle in the following order: $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$. Points $A_3$, $B_3$ and $C_3$ are the midpoints of the arc $A_1A_2$, $B_1B_2$ and $C_1C_2$ respectively. Lines $A_2A_3$ and $B_1B_3$ intersect in $C_4$, lines $B_2B_3$ and $C_1C_3$ – in $A_4$, and lines $C_2C_3$ and $A_1A_3$ – in $B_4$. Prove that the segments $A_3A_4$, $B_3B_4$ and $C_3C_4$ intersect in one point. (cf. Fig. 24)

25. (Bulgarian IMO Test ’08) In $\triangle ABC$ let $AM$ ($M \in BC$) be a median and let $CC_1$ ($C_1 \in AB$) and $BB_1$ ($B_1 \in AC$) be two altitudes. The line through $A$ perpendicular to $AM$ intersects lines $BB_1$ and $CC_1$ in points $E$ and $F$, respectively. Denote by $k$ the circumcircle of $\triangle EFM$. Let $k_1$ and $k_2$ be circles tangent to both $EF$ and to the arc $EF$ on $K$ not containing $M$. If $P$ and $Q$ are the intersection points of $k_1$ and $k_2$, prove that points $P$, $Q$, and $M$ lie on a line.
26. (IMO ’08, Spain) An acute-angled \( \triangle ABC \) has orthocenter \( H \). The circle passing through \( H \) with center the midpoint of \( BC \) intersects the line \( BC \) at \( A_1 \) and \( A_2 \). Similarly, the circle passing through \( H \) with center the midpoint of \( CA \) intersects the line \( CA \) at \( B_1 \) and \( B_2 \), and the circle passing through \( H \) with center the midpoint of \( AB \) intersects the line \( AB \) at \( C_1 \) and \( C_2 \). Show that \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) lie on a circle.

27. (BAMO ’06) Given \( \triangle ABC \), let \( A_1, B_1, \) and \( C_1 \) be points on sides \( BC, CA, \) and \( AB \), respectively, such that lines \( AA_1, BB_1, \) and \( CC_1 \) intersect in one point \( P \). Prove that \( P \) is the centroid of \( \triangle ABC \) if and only if it is the centroid of \( \triangle A_1B_1C_1 \).

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