Berkeley Math Circle — Jan. 27, 2009

BAMO Practice for Beginners

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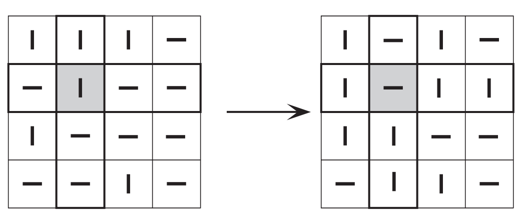
The "easier" BAMO problems, 1999–2008

From 1999-2007, BAMO had five problems, in order of difficulty. Starting with 2008, BAMO now has 7 problems: BAMO-8 consists of problems 1–4, and is open only to students in grade 8 and below. BAMO-12 consists of problems 3–7, with difficulty level comparable to the "old" BAMO. Hence, starting with 2008, problems 1 and 2 are designed to be rather easy. Not that any BAMO problem is truly easy...

1999.1 Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors. (The proper divisors of a positive integer n are all positive integers other than 1 and n which divide n. For example, the proper divisors of 14 are 2 and 7.)

1999.2 Let C be a circle in the xy-plane with center on the y-axis and passing through $A = (0,a)$ and $B = (0,b)$ with $0 < a < b$. Let P be any other point on the circle, let Q be the intersection of the line through P and A with the x-axis, and let $O = (0,0)$. Prove that $\angle BQP = \angle BOP$.

1999.3 A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see diagram). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)



before switching shaded key — *after switching shaded key*

2000.1 Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22 + 15$ represents the number 37 in the desired way.)

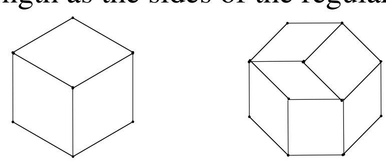
2001.1 Each vertex of a regular 17-gon is colored red, blue, or green in such a way that no two adjacent vertices have the same color. Call a triangle “multicolored” if its vertices are colored red, blue, and green, in some order. Prove that the 17-gon can be cut along nonintersecting diagonals to form at least two multicolored triangles.

(A *diagonal* of a polygon is a a line segment connecting two nonadjacent vertices. Diagonals are called *nonintersecting* if each pair of them either intersect in a vertex or do not intersect at all.)

2001.2 Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y and Z are collinear (lie on the same line).

2002.1 Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

2002.2 In the illustration, a regular hexagon and a regular octagon have been tiled with rhombuses. In each case, the sides of the rhombuses are the same length as the sides of the regular polygon.



(a) Tile a regular decagon (10-gon) into rhombuses in this manner.

(b) Tile a regular dodecagon (12-gon) into rhombuses in this manner.

(c) How many rhombuses are in a tiling by rhombuses of a 2002-gon? Justify your answer.

2002.3 A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. The winner is the player who makes a move that causes all stacks to be of height 1 or 2. For which starting values of n does the player who goes first win, assuming best play by both players?

2003.1 An integer is a *perfect* number if and only if it is equal to the sum of all of its divisors except itself. For example, 28 is a perfect number since $28 = 1+2+4+7+14$.

Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$, where n is a positive integer. An integer is a *factorial* if and only if it is equal to $n!$ for some positive integer n. For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.

Find all perfect numbers greater than 1 that are also factorials.

2003.2 Five mathematicians find a bag of 100 gold coins in a room. They agree to split up the coins according to the following plan:

- The oldest person in the room proposes a division of the coins among those present. (No coin may be split.) Then all present, including the proposer, vote on the proposal.
- If at least 50% of those present vote in favor of the proposal, the coins are distributed accordingly and everyone goes home. (In particular, a proposal wins on a tie vote.)
- If fewer than 50% of those present vote in favor of the proposal, the proposer must leave the room, receiving no coins. Then the process is repeated: the oldest person remaining proposes a division, and so on.
- There is no communication or discussion of any kind allowed, other than what is needed for the proposer to state his or her proposal, and the voters to cast their vote.

Assume that each person is equally intelligent and each behaves optimally to maximize his or her share. How much will each person get?

2003.3 A *lattice point* is a point (x, y) with both x and y integers. Find, with proof, the smallest n such that every set of n lattice points contains three points that are the vertices of a triangle with integer area. (The triangle may be *degenerate*, in other words, the three points may lie on a straight line and hence form a triangle with area zero.)

2004.1 A *tiling* of the plane with polygons consists of placing the polygons in the plane so that interiors of polygons do not overlap, each vertex of one polygon coincides with a vertex of another polygon, and no point of the plane is left uncovered. A *unit* polygon is a polygon with all sides of length one.

It is quite easy to tile the plane with infinitely many unit squares. Likewise, it is easy to tile the plane with infinitely many unit equilateral triangles.

(a) Prove that there is a tiling of the plane with infinitely many unit squares and infinitely many unit equilateral triangles in the same tiling.

(b) Prove that it is impossible to find a tiling of the plane with infinitely many unit squares and finitely many (and at least one) unit equilateral triangles in the same tiling.

2004.2 A given line passes through the center O of a circle. The line intersects the circle at points A and B. Point P lies in the exterior of the circle and does not lie on the line AB. Using only an unmarked straightedge, construct a line through P, perpendicular to the line AB. Give complete instructions for the construction and prove that it works.

2005.1 An integer is called *formidable* if it can be written as a sum of distinct powers of 4, and *successful* if it can be written as a sum of distinct powers of 6. Can 2005 be written as a sum of a formidable number and a successful number? Prove your answer.

2005.2 Prove that if two medians in a triangle are equal in length, then the triangle is isosceles. (Note: A median in a triangle is a segment which connects a vertex of the triangle to the midpoint of the opposite side of the triangle.)

2005.3 Let $n \geq 12$ be an integer, and let $P_1, P_2, \ldots, P_n, Q$ be distinct points in the plane. Prove that for some i, at least $n/6 - 1$ of the distances

$$P_1P_i, P_2P_i, \ldots, P_{i-1}P_i, P_{i+1}P_i, \ldots, P_nP_i$$

are less than P_iQ.

2006.1 All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, n columns and n rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

(a) Every student must move to a new chair.

(b) A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically.

(Note that the rules above allow two students in adjacent chairs to exchange places.)

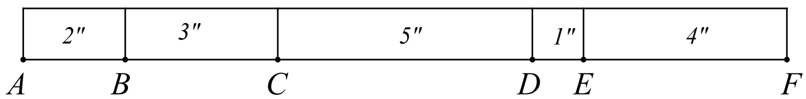
Show that this procedure can be done if n is even, and cannot be done if n is odd.

2006.2 Since $24 = 3 + 5 + 7 + 9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

(a) Can 2005 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

(b) Can 2006 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

2007.1 A 15-inch-long stick has four marks on it, dividing it into five segments of length $1, 2, 3, 4$, and 5 inches (although not neccessarily in that order) to make a "ruler." Here is an example.



Using this ruler, you could measure 8 inches (between the marks B and D) and 11 inches (between the end of the ruler at A and the mark at E), but there's no way you could measure 12 inches.

Prove that it is impossible to place the four marks on the stick such that the five segments have length $1,2,3,4$, and 5 inches, and such that every integer distance from 1 inch through 15 inches could be measured.

2007.2 The points of the plane are colored in black and white so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

2008.1 Call a year *ultra-even* if all of its digits are even. Thus $2000, 2002, 2004, 2006$, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

(a) In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.

(b) What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

2008.2 Consider a 7×7 chessboard that starts out with all the squares white. We start painting squares black, one at a time, according to the rule that after painting the first square, each newly painted square must be adjacent along a side to only the square just previously painted. The final figure painted will be a connected "snake" of squares.

(a) Show that it is possible to paint 31 squares.

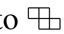
(b) Show that it is possible to paint 32 squares.

(c) Show that it is possible to paint 33 squares.

2008.3 A triangle (with non-zero area) is constructed with the lengths of the sides chosen from the set

$$\{2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}.$$

Show that this triangle must be isosceles (A triangle is *isosceles* if it has at least two sides the same length.)

Determine the greatest number of figures congruent to  that can be placed in a 9×9 grid (without overlapping), such that each figure covers exactly 4 unit squares.

2008.4 N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a Californian team Alcatraz is the unique Californian champion (Alcatraz has won more games against Californian teams than any other team from California). However, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation!

What is the smallest possible value for N?