PELL'S EQUATION AND CONTINUED FRACTIONS

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1. Introduction

The topic of *Pell's Equation and Continued Fractions* has a long history spanning thousands of years and various cultures. This paper will collect a number of problems to demonstrate the variety of techniques that have been devised. This will connect the geometry with algebra and the incommensurable with the commensurable. The name Pell's equation is a misnomer. Euler mistakenly thought Pell had solved it and began calling it "Pell's Equation", making it famous. When it was discovered that Pell did not solve it, it was too late to change it. The equation is of the form $x^2 - dy^2 = \pm 1$ where d is not a perfect square and the solutions being sought are positive integers.

2. Euclid and Archimedes

Proclus in his commentary on Plato's *Republic* indicates that Propositions 9 and 10 of Book II of Euclid's *The Elements* are theorems providing an algorithm for successive approximations of $\sqrt{2}$ from the side and diagonal of a procession of isosceles right triangles. In *Republic* Plato refers, following the Pythagoreans, to a diagonal of the square on the straight line of side 5 using the "inexpressible" $\sqrt{50}$ and the "expressible" approximation $\sqrt{50-1} = 7$. This would suggest that the process was known to the Pythagoreans.

Exercise 1. (Euclid Book II Proposition 9) If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the squares on the half and of the square on the straight line between the points of section. (In more modern language: Let M be the midpoint of $\overline{AB}$ and C between M and B. Prove $2AM^2 + 2MC^2 = AC^2 + CB^2$.

Exercise 2. Use the previous exercise to derive the algebraic identity

$$p^2 - 2q^2 = 2(p+q)^2 - (p+2q)^2.$$

Exercise 3. Note that $3^2 - 2 \cdot 2^2 = 1$ and use the previous exercise to find another solution to $x^2 - dy^2 = 1$. Then use that solution to find another solution.

Exercise 4. Use the Euclidean algorithm (Book VII Proposition 1) to find all the solutions in integers to the equation $707x + 500y = 1$.

Exercise 5. Use the Euclidean algorithm to express $\frac{707}{500}$ as a continued fraction, i.e.,

$$\frac{8}{5} = 1 + \cfrac{1}{1 + \cfrac{1}{2}}$$

Exercise 6. Find continued fractions for $\sqrt{2}$ and $\sqrt{3}$. Compare the convergents of $\sqrt{2}$ and the convergents of $\frac{707}{500}$.

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Exercise 7. In his treatise *Measurement of the Circle*, Archimedes requires a rational expression which is a close approximation and an upper bound for $\sqrt{3}$. Without any comment on how he found the fraction he produces $\frac{1351}{780}$. Compute the twelfth convergent of $\sqrt{3}$ from the continued fraction and compare the result.

Exercise 8. In his famous "Cattle Problem" Archimedes poses the problem of counting the number of oxen in the sun. First solve seven linear equations in eight unknowns:

$$W = (\frac{1}{2}+\frac{1}{3})B + Y \qquad B = (\frac{1}{4}+\frac{1}{5})D + Y \qquad D = (\frac{1}{4}+\frac{1}{5})W + Y$$

$$w = (\frac{1}{3}+\frac{1}{4})b \qquad b = (\frac{1}{4}+\frac{1}{5})d \qquad d = (\frac{1}{5}+\frac{1}{6})y \qquad y = (\frac{1}{6}+\frac{1}{7})w$$

Exercise 9. Find which triangular numbers are perfect squares, i.e., find positive integers n and m such that $\frac{n(n+1)}{2} = m^2$.

Exercise 10. The real cattle problem now begins by further stipulating that the sum $W + B$ must be a perfect square and $D + Y$ must be a triangular number. The problem was not solved until the late 1800's and the actual number (all 206,545 digits) was not produced until 1965 when computers were available. If $D + Y = 11507447k$ and $W + B = 17826996k = 4 \cdot 3 \cdot 11 \cdot 29 \cdot 4657k$, then $k = 3 \cdot 11 \cdot 29 \cdot 4657m^2 = 4456749m^2$. Show that the condition that $D + Y$ must be a triangular number leads to solving the equation $x^2 - 410286423278424\, m^2 = 1$. (Noticing that $410286423278424 = 2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 \cdot 4657^2$ this can be simplified to finding solutions to $x^2 - 4729494\, m_1^2 = 1$.)

3. Brahmagupta and Bhaskara

Exercise 11. Around 650 C.E. the Indian mathematician Brahmagupta posed the following problem: "Making the square of the residue of signs and minutes . . . with one added to the product . . . an exact square. A person solving this problem within a year is a mathematician." The problem is to solve $92y^2 + 1 = x^2$. Brahmagupta had a cycling method for solving such problems which started with an approximate answer, but sometimes this led to non-integral rational solutions. Show that you are a mathematician.

Exercise 12. About 500 years later the great Indian mathematician Bhaskara added more rules to the cycling method to find the general solution of the problem. Using the identity $(a^2 - 67b^2)(c^2 - 67d^2) = (ac + 67bd)^2 - 67(ad + bc)^2$ he was able to solve the equation $x^2 - 67\,y^2 = 1$. He also solved the much harder $x^2 - 61\,y^2 = 1$ whose smallest solution for x has nine digits. Can you solve this?

4. Fermat and Wallis

Exercise 13. In 1657, Fermat challenged mathematicians, and in particular, English mathematicians as follows: "To arithmeticians . . . I propose the following theorem to be proved or problems to be solved. If they succeed in finding the proof or solution, they will admit that the questions of this kind are not inferior to the more celebrated questions of geometry in respect of beauty, difficulty, or method of proof. *Given any number whatever which is not a square, there are also given an infinite number of squares such that, if the square is multiplied into the given number and unity is added to the product, the result is a square.*" As examples he proposed using 149, 109 (15 digits) and 433. He also challenged the French mathematician Frénicle with the Bhaskara number 61. Will you take up the challenge? Solve $x^2 - 109y^2 = 1$, $x^2 - 149y^2 = 1$, and $x^2 - 433y^2 = 1$. Actually, $x^2 - 421y^2 = 1$ is harder with a smallest solution for x having 34 digits.

Exercise 14. The English mathematicians John Wallis and Lord Brouncker responded to Fermat's challenge with a general method to which Fermat then indicated that he had solutions to the equation for non-squares from 2 to 150 and perhaps the English might extend the table to 200 or at least solve it for 151. Brouncker replied "Within the space of an hour or two at most this morning, according to the method therein delivered, I found that $313 \cdot 7170685^2 - 1 = 126862368^2$, so that $313(2 \cdot 7170685 \cdot 126862368)^2 + 1 = 313 \cdot 1819380158564160^2 + 1 = 32188120829134849^2$, which I thought fit to present to you that M. Frénicle may thence perceive that nothing is wanting to the perfect solution of that problem." Needless to say, Fermat and Frénicle were very impressed. Explain how the first equation with "-1" leads to the final solution in general. (How is 32188120829134849 constructed from the first equation?)

5. Euler and Lagrange

In 1732, Euler in a letter to Goldbach mentions that the solution to $x^2 - d\,y^2 = 1$ is crucial in understanding the solutions to the general second degree equation. (It is Euler who first proves the claim of Fermat in the previous century that primes of the form $4k+1$ can be expressed uniquely as a sum of two squares.) In the letter, he goes on to state that Wallis and Fermat discussed them and at this point makes his first misreading of events confusing the solution by Lord Brouncker with the name Pell. When he returned to the subject later in his influential book *Elements of Algebra* in 1770 he makes the same error and so the name of Pell will remain entwined with this equation forever. In this book Euler lays out his solution technique using continued fractions which he developed throughout the forty years he thought about the problem. The actual problem of showing that all of the solutions for any non-square d were found was not proved by Euler, but a supplement added to Euler's book, prepared by Lagrange, remedied the situation with a complete proof that there will always be a solution to $x^2 - dy^2 = 1$ and every solution can be found from the smallest solution. In the next century, Gauss stated "the treatise of Lagrange grasps the problem in its entire generality and in this connection leaves nothing to be desired."

Exercise 15. The purely periodic continued fraction $\alpha = \langle \overline{a_1, a_2 \cdots, a_n} \rangle$ is greater than 1 and is the positive root of a quadratic equation with integral coefficients. The continued fraction with the period reversed, $\beta = \langle \overline{a_n, a_{n-1} \cdots, a_1} \rangle$, is the conjugate root of the quadratic equation satisfied by α.

Exercise 16. The converse of the statements in the previous exercise are also true. This implies that $\sqrt{d}$ is will eventually become periodic.

Exercise 17. Find the periodic continued fraction for $\sqrt{a^2+1}$.

Exercise 18. Compute the numbers $(3+2\sqrt{2})^n$ for $n = 1, 2, 3, 4$. How do the coefficients of the rational and irrational parts of your answers relate to the the convergents for $\sqrt{2}$.

Exercise 19. Show that if (x_1, y_1) is the solution of $x^2 - dy^2 = 1$ having the smallest x value, then the k^{th} solution is (x_k, y_k) where $(x_1 + y_1\sqrt{d})^k = x_k + y_k\sqrt{d}$. Showing that all the powers lead to solutions is not difficult. Proving that there are no other solutions can be done using Fermat's "method of descent", i.e., assume that there is such a solution, consider the smallest, and then show that this leads to a smaller solution.

Exercise 20. Show that $|x - y\sqrt{d}| < \frac{1}{y}$ is satisfied by an infinite number of pairs of positive integers (x, y).

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Closing, here are a few remarkable continued fractions. Lord Brouncker, 1658.

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \ddots}}}}}$$

Euler, 1737 (Euler proved that e is irrational and Hermite proved that e is transcendental).

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \ddots}}}}}}$$

This continues with $1, 6, 1, 1, 8, 1, \ldots, 1, 2n, 1, \ldots$.

Euler in 1737 and Lambert, who first proved π is irrational, produced this continued fraction.

$$\frac{e^{1/s} + 1}{e^{1/s} - 1} = 2s + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \cfrac{1}{18s + \ddots}}}}$$

Finally, a continued fraction of Ramanujan that relates π, e, and the golden ratio ϕ.

$$(\sqrt{\phi + 2} - \sqrt{\phi})e^{2\pi/5} = 1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \cfrac{e^{-8\pi}}{1 + \ddots}}}}$$

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