Multiplicative Functions and Möbius Inversion Formula

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1. Multiplicative Functions. Overview

Definition 1. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be *arithmetic.*

In this section we discuss the set $\mathcal{M}$ of *multiplicative* functions, which is a subset of the set $\mathcal{A}$ of arithmetic functions. Why this subset is so special can be explained by the fact that it is usually easier to calculate explicit formulas for multiplicative functions, and not so easy to do this for arbitrary arithmetic functions. [1]

Definition 2. An arithmetic function $f(n) : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if for any relatively prime $n, m \in \mathbb{N}$:

$$f(mn) = f(m)f(n).$$

Examples. Let $n \in \mathbb{N}$. Define functions $\tau, \sigma, \pi : \mathbb{N} \to \mathbb{N}$ as follows:

- $\tau(n)$ = the number of all natural divisors of $n = \#\{d > 0 \mid d|n\}$;
- $\sigma(n)$ = the sum of all natural divisors of $n = \sum_{d|n} d$;
- $\pi(n)$ = the product of all natural divisors of $n = \prod_{d|n} d$.

As we shall see below, τ and σ are multiplicative functions, while π is not. From now on we shall write the prime decomposition of $n \in \mathbb{N}$ as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ for distinct primes p_i and $\alpha_i > 1$. It is easy to verify the following properties of multiplicative functions:

Lemma 1. *If f is multiplicative, then either $f(1) = 1$ or $f \equiv 0$. Further, if $f_1, f_2, ..., f_k$ are multiplicative, then the usual product $f_1 f_2 \cdots f_n$ is also multiplicative.*

Using the prime decomposition of $n \in \mathbb{N}$, derive the following representations of τ, σ and π.

Lemma 2. *The functions $\tau(n), \sigma(n)$ and $\pi(n)$ are given by the formulas:*

$$\tau(n) = \prod_{i=1}^{r} (\alpha_i + 1),\ \sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1},\ \pi(n) = n^{\frac{1}{2}\tau(n)}.$$

Conclude that τ and σ are multiplicative, while π is not.

Examples. The following are further examples of well-known multiplicative functions.

- $\mu(n)$, the Möbius function;
- $e(n) = \delta_{1,n}$, the Dirichlet identity in $\mathcal{A}$;
- $I(n) = 1$ for all $n \in \mathbb{N}$;
- $id(n) = n$ for all $n \in \mathbb{N}$.

Taking the sum-functions of these, we obtain the relations: $S_\mu = e$, $S_e = I$, $S_I = \tau$, and $S_{id} = \sigma$. These examples suggest that the sum-function is multiplicative, provided the original function is too. In fact,

[1] But on a deeper level, $\mathcal{M}$ is special partly because it is closed under the Dirichlet product in $\mathcal{A}$. For this one has to wait until after Section 2.

Theorem 1. *$f(n)$ is multiplicative iff its sum-function $S_f(n)$ is multiplicative.*

Proof: Let $f(n)$ be multiplicative, and let $x, y \in \mathbb{N}$ such that $(x,y)=1$. Further, let $x_1, x_2, ..., x_k$ and $y_1, y_2, ..., y_m$ be all divisors of x and y, respectively. Then $(x_i, y_j) = 1$, and $\{x_i y_j\}_{i,j}$ are all divisors of xy.

$$\Rightarrow \; S_f(x) \cdot S_f(y) = \sum_{i=1}^{k} f(x_i) \sum_{j=1}^{m} f(y_j) = \sum_{i,j} f(x_i) f(y_j) = \sum_{i,j} f(x_i y_j) = S_f(xy).$$

Hence $S_f(n)$ is multiplicative.

Conversely, if $S_f(n)$ is multiplicative, then let $n_1, n_2 \in \mathbb{N}$ such that $(n_1, n_2) = 1$ and $n = n_1 n_2$. We will prove by induction on n that $f(n_1 n_2) = f(n_1) f(n_2)$. The statement is trivial for $n = 1$: $f(1) = S_f(1) (= 1$ or $0.)$ Assume that it is true for all $m_1 m_2 < n$. Then for our n_1, n_2 we have:

$$S_f(n_1 n_2) = \sum_{d_i | n_i} f(d_1 d_2) = \sum_{\substack{d_i | n_i \\ d_1 d_2 < n}} f(d_1 d_2) + f(n_1 n_2) = \sum_{\substack{d_i | n_i \\ d_1 d_2 < n}} f(d_1) f(d_2) + f(n_1 n_2).$$

On the other hand,

$$S_f(n_1) S_f(n_2) = \sum_{d_1 | n_1} f(d_1) \sum_{d_2 | n_2} f(d_2) = \sum_{\substack{d_i | n_i \\ d_1 d_2 < n}} f(d_1) f(d_2) + f(n_1) f(n_2).$$

Since $S_f(n_1 n_2) = S_f(n_1) S_f(n_2)$, equating the above two expressions and canceling appropriately, we obtain $f(n_1 n_2) = f(n_1) f(n_2)$. This completes the induction step, and shows that $f(n)$ is indeed multiplicative. $\square$

Corollary 1. *The sum-function $S_f(n)$ of a multiplicative function $f(n)$ is given by the formula:*

$$S_f(n) = \prod_{i=1}^{r} \left(1 + f(p_i) + f(p_i^2) + \cdots + f(p_i^{\alpha_i})\right).$$

2. Dirichlet Product and Möbius Inversion

Consider the set $\mathcal{A}$ of all arithmetic functions, and define the *Dirichlet product* of $f, g \in \mathcal{A}$ as:

$$f \circ g(n) = \sum_{d_1 d_2 = n} f(d_1) f(d_2).$$

Note that $f \circ g$ is also arithmetic, and that the product $\circ$ is commutative, and associative:

$$(f \circ g) \circ h(n) = f \circ (g \circ h)(n) = f \circ g \circ h(n) = \sum_{d_1 d_2 d_3 = n} f(d_1) f(d_2) f(d_3).$$

With respect to the Dirichlet product, the identity element $e \in \mathcal{A}$ is easy to find:

$$e(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Indeed, check that $e \circ f = f \circ e = f$ for any $f \in \mathcal{A}$. If we were working with the usual product of functions $f \cdot g(n) = f(n) g(n)$, then the "identity" element would have been

$$I(n) = 1 \text{ for all } n \in \mathbb{N},$$

because $I \cdot f = f$ for all functions f. In the set $\mathcal{A}$, however, I is certainly *not* the identity element, but has the nice property of transforming each function f into its so-called *sum-function* S_f. Define

$$S_f(n) = \sum_{d|n} f(d),$$

to be the *sum-function* of $f \in \mathcal{A}$, and note that S_f is also arithmetic. Check that:

$$I \circ f = f \circ I = S_f \text{ for all } f \in \mathcal{A}.$$

It is interesting to find the *Dirichlet inverse* $g \in \mathcal{A}$ of this simple function I in our set $\mathcal{A}$, i.e. such that $g \circ I = I \circ g = e$. This naturally leads to the introduction of the so-called *Möbius function* μ:

Definition 2. The *Möbius function* $\mu : \mathbb{N} \to \mathbb{C}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not square-free} \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r,\ p_j - \text{distinct primes.} \end{cases}$$

Lemma 3. *The Dirichlet inverse of I is the Möbius function $\mu \in \mathcal{A}$.*

Proof: The lemma means that $\mu \circ I = e$, i.e.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1. \end{cases} \tag{1}$$

This follows easily from the definition of μ. Indeed, for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} > 1$ we have

$$\sum_{d|n} \mu(d) = \sum_{d|n, d-\text{sq.free}} \mu(d) = \mu(1) + \sum_{k=1}^{r} \sum_{1 \le i_1 < \cdots < i_k \le r} \mu(p_{i_1} \cdots p_{i_k}).$$

Here the last sum runs over all square-free divisors of n. For combinatorial reasons,

$$\sum_{d|n} \mu(d) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k = (1-1)^r = 0. \quad \square$$

To summarize, we have shown that the Dirichlet inverse of the function $I(n)$ is the Möbius function $\mu(n)$: $\mu \circ I = I \circ \mu = e$. Unfortunately, not all arithmetic functions have Dirichlet inverses in $\mathcal{A}$. In fact, show that

Lemma 4. *An arithmetic function f has a Dirichlet inverse in $\mathcal{M}$ iff $f(1) \neq 0$.*[2]

The "right" notion to replace "inverses in $\mathcal{A}$" turns out to be "sum-functions", which is the idea of the Möbius inversion theorem.

Theorem 2 (Möbius inversion theorem)**.** *Any arithmetic function $f(n)$ can be expressed in terms of its sum-function $S_f(n) = \sum_{d|n} f(d)$ as*

$$f(n) = \sum_{d|n} \mu(d) S_f(\frac{n}{d}).$$

[2]For those who care about group theory interpretations, this implies that $\mathcal{M}$ is *not* a group with the Dirichlet product, but the subset $\mathcal{M}'$ of it consisting of all arithmetic functions f with $f(1) \neq 0$ is a group.

Proof: The statement is nothing else but the Dirichlet product $f = \mu \circ S_f$ in $\mathcal{A}$:

$$\mu \circ S_f = \mu \circ (I \circ f) = (\mu \circ I) \circ f = e \circ f = f. \quad \Box$$

Note: Here is a more traditional proof of the Möbius Inversion Formula:

$$\begin{aligned}\sum_{d|n} \mu(d) S_f(\frac{n}{d}) &= \sum_{d|n} \mu(\frac{n}{d}) S_f(d) = \sum_{d|n} \mu(\frac{n}{d}) \sum_{d_1|d} f(d_1) \\ &= \sum_{d_1|n} f(d_1) \sum_{d_1|d|n} \mu(\frac{n}{d}) = \sum_{d_1|n} f(d_1) \sum_{d_2|m} \mu(\frac{m}{d_2}),\end{aligned}$$

where $m = n/d_1$, $d_2 = d/d_1$. By property (1), the second sum is non-zero only when $m = 1$, i.e. $d_1 = n$, and hence the whole expression equals $f(n)$. $\Box$

This proof, however, hides the product structure of $\mathcal{A}$ under the Dirichlet product. It is natural to asks the opposite question: given the Möbius relation $f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$ for two arithmetic functions f and g, can we deduce that g is the sum-function S_f of f? The answer should be obvious from the product structure of $\mathcal{A}$:

$$f = \mu \circ g \;\Rightarrow\; I \circ f = I \circ (\mu \circ g) \;\Rightarrow\; S_f = (I \circ \mu) g = e \circ g = g,$$

and indeed, g is the sum-function of f. If you prefer more traditional proofs, you can recover one from the above by recalling that "multiplying f by I" is the same as taking the sum-function of f.

Corollary 2. *For two arithmetic functions f and g we have the following equivalence:*

$$g(n) = \sum_{d|n} f(d) \;\Leftrightarrow\; f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}).$$

This incidentally shows that every arithmetic function g is the sum-function of another arithmetic function f: simply define f by the second formula as $f = \mu \circ g$.

Notice that Theorem 1 does not use the Dirichlet product at all, but instead it hides a more general fact. For an arithmetic function f, its sum-function is $S_f = I \circ f$, and by Möbius inversion, $f = \mu \circ S_f$. Thus, Theorem 1 simply proves that if $f \in \mathcal{M}$, then the product of the two multiplicative functions I and f is also multiplicative, and that if $S_f \in \mathcal{M}$, then the product of the two multiplicative functions S_f and μ is also multiplicative. This naturally suggest the more general

Theorem 3. *The set $\mathcal{M}$ of multiplicative functions is closed under the Dirichlet product: $f, g \in \mathcal{M} \Rightarrow f \circ g \in \mathcal{M}$.*

Proof: Let $f, g \in \mathcal{M}$, $(a, b) = 1$, and $h = f \circ g$. Then

$$\begin{aligned}h(a)h(b) &= \big[f \circ g(a)\big] \cdot \big[f \circ g(b)\big] = \sum_{d_1|a} f(d_1) g(\frac{a}{d_1}) \sum_{d_2|b} f(d_2) g(\frac{b}{d_2}) \\ &= \sum_{d_1|a, d_2|b} f(d_1) f(d_2) g(\frac{a}{d_1}) g(\frac{b}{d_2}) = \sum_{d_1|a, d_2|b} f(d_1 d_2) g(\frac{ab}{d_1 d_2}) \\ &= \sum_{d|ab} f(d) g(\frac{ab}{d}) = f \circ g(ab) = h(ab).\end{aligned}$$

Thus, h is also multiplicative, so that $f \circ g \in \mathcal{M}$. $\Box$

For the group theory fans: find the explicit Dirichlet inverse of any multiplicative function $f \not\equiv 0$, and conclude

Lemma 5. *The set* $\mathcal{M} - \{f \equiv 0\}$ *is a group under the Dirichlet product.*

3. Warm-up Problems

Problem 1. Find $m, n \in \mathbb{N}$ such that they have no prime divisors other than 2 and 3, $(m, n) = 18$, $\tau(m) = 21$, and $\tau(n) = 10$.

Problem 2. Find $n \in \mathbb{N}$ such that one of the following is satisfied: $\pi(n) = 2^3 \cdot 3^6$, $\pi(n) = 3^{30} \cdot 5^{40}$, $\pi(n) = 13 \cdot 31$, or $\tau(n) = 13 \cdot 31$.

Problem 3. Define $\sigma_k(n) = \sum_{d|n} d^k$. Thus, $\sigma_0(n) = \tau(n)$ and $\sigma_1(n) = \sigma(n)$. Prove that $\sigma_k(n)$ is multiplicative for all $k \in \mathbb{N}$, and find a formula for it.

Problem 4. Show that $\tau(n)$ is odd iff n is a perfect square, and that $\sigma(n)$ is odd iff n is a perfect square or twice a perfect square.

Problem 5. If $f(n)$ is multiplicative, $f \not\equiv 0$, then show $\sum_{d|n} \mu(d)f(d) = \prod_{i=1}^{r} \big(1 - f(p_i)\big)$.

Problem 6. If $f(n)$ is multiplicative, then show that $h(n) = \sum_{d|n} \mu(\frac{n}{d})f(d)$ is also multiplicative. Conclude that every multiplicative function is the sum-function of another multiplicative function.

4. The Euler Function $\varphi(n)$

Definition 4. The *Euler function* $\varphi(n)$ assigns to each natural number n the number of the integers d between 1 and n which are relatively prime to n (set $\varphi(1) = 1$.)

Lemma 6. $\varphi(n)$ *is multiplicative.*

Proof: Consider the table of all integers between 1 and ab, where $a, b \in \mathbb{N}$, $(a, b) = 1$.

1	2	$\cdots$	i	$\cdots$	$a-1$	a
$a+1$	$a+2$	$\cdots$	$a+i$	$\cdots$	$2a-1$	$2a$
$2a+1$	$2a+2$	$\cdots$	$2a+i$	$\cdots$	$3a-1$	$3a$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$ja+1$	$ja+2$	$\cdots$	$ja+i$	$\cdots$	$(j+1)a-1$	$(j+1)a$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(b-2)a+1$	$(b-2)a+2$	$\cdots$	$(b-2)a+i$	$\cdots$	$(b-1)a-1$	$(b-1)a$
$(b-1)a+1$	$(b-1)a+2$	$\cdots$	$(b-1)a+i$	$\cdots$	$ba-1$	ba

Note that if $ja + i$ is relatively prime with a, then all numbers in its (i-th) column will be relatively prime with a. In the first row there are exactly $\varphi(a)$ numbers relatively prime with a, so their $\varphi(a)$ columns will be all numbers in the table which are relatively prime with a.

As for b, each column is a system of remainders modulo b because $(a, b) = 1$ (check that in each column the b integers are distinct modulo b.) Thus, in each column we have precisely $\varphi(b)$ relatively prime integers to b. Hence, the total number of elements in this table relatively prime to ab is exactly $\varphi(ab) = \varphi(a)\varphi(b)$, and $\varphi(n)$ is multiplicative. $\square$

We can use multiplicativity of the Euler function to derive a formula for $\varphi(n)$. For a prime p, all numbers $d \in [1, p^k]$ such that $(d, p^k) \neq 1$ are exactly those d divisible by p: $d = p \cdot c$ with $c = 1, 2, ..., p^{k-1}$, i.e. p^{k-1} in number. Hence $\varphi(p^k) = p^k - p^{k-1}$. Therefore,

$$\varphi(n) = \prod_{i=1}^{r} \varphi(p_i^{\alpha_i}) = \prod_{i=1}^{r} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = \prod_{i=1}^{r} p_i^{\alpha_i}(1 - \frac{1}{p_i}) = n(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_r}).$$

Lemma 7. *The sum-function $S_\varphi(n)$ of the Euler function $\varphi(n)$ satisfies:*

$$\sum_{d|n} \varphi(n) = n.$$

Proof: The multiplicativity of the Euler function $\varphi(n)$ implies that $S_\varphi(n)$ is multiplicative, so that $S_\varphi(n) = S_\varphi(p_1^{\alpha_1}) \cdots S_\varphi(p_r^{\alpha_r})$. This reduces the problem to a prime power $n = p^a$, which together with $\varphi(p^j) = p^j - p^{j-1}$ easily implies $S_\varphi(p^a) = p^a$. □

Remark: We can prove the lemma also directly by considering the set of fractions

$$\left\{ \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n} \right\} = \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_{n-1}}{b_{n-1}}, \frac{a_n}{b_n} \right\}$$

where $(a_i, b_i) = 1$. The set $\mathcal{D}$ of all denominators $\{b_i\}$ is precisely the set of all divisors of n. Given a divisor d of n, d appears in the set $\mathcal{D}$ exactly $\varphi(d)$ times:

$$\frac{a_i}{b_i} = \frac{a_i}{d} = \frac{a_i \frac{n}{d}}{n},$$

where $(a_i, d) = 1$ and $1 \leq a_i \leq d$. Counting the elements in $\mathcal{D}$ in two different ways implies $n = \sum_{d|n} \varphi(d)$. □

We can use this Remark to show again multiplicativity of the Euler function: indeed, this follows from the fact that its sum-function $S_\varphi(n) = n$ is itself obviously multiplicative.

5. Warm-up Problems

Problem 7. Show that $\varphi(n^k) = n^{k-1}\varphi(n)$ for all $n, k \in \mathbb{N}$.

Problem 8. Solve the following equations:
- $\varphi(2^x 5^y) = 80$.
- $\varphi(n) = 12$.
- $\varphi(n) = 2n/3$.
- $\varphi(n) = n/2$.
- $\varphi(\varphi(n)) = 2^{13}3^3$.

Problem 9. Show that
- $\varphi(n)\varphi(m) = \varphi((n, m))\varphi([n, m])$;
- $\varphi(nm)\varphi((n, m)) = (n, m)\varphi(n)\varphi(m)$.

6. Applications to Problems

Problem 10. For two sequences of complex numbers $\{a_0, a_1, ..., a_n, ...\}$ and $\{b_0, b_1, ..., b_n, ...\}$ show that the following relations are equivalent:

$$a_n = \sum_{k=0}^{n} b_k \text{ for all } n \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{k+n} a_k \text{ for all } n.$$

Problem 11. Solve the equation $\varphi(\sigma(2^n)) = 2^n$.

Problem 12. Let $f(x) \in \mathbb{Z}[x]$ and let $\psi(n)$ be the number of values $f(j),\ j = 1, 2, ..., n$, such that $(f(j), n) = 1$. Show that $\psi(n)$ is multiplicative and that $\psi(p^t) = p^{t-1}\psi(p)$. Conclude that

$$\psi(n) = \prod_{p|n} \psi(p)/p.$$

Problem 13. Find closed expressions for the following sums:

- $\sum_{d|n} \mu(d)\tau(d)$
- $\sum_{d|n} \mu(d)\varphi(d)$
- $\sum_{d|n} \mu(d)\sigma(d)$
- $\sum_{d|n} \mu^2(d)\varphi^2(d)$
- $\sum_{d|n} \mu(d)\tau(\frac{n}{d})$
- $\sum_{d|n} \frac{\mu(d)}{\varphi(d)}$
- $\sum_{d|n} \mu(d)\sigma(\frac{n}{d})$
- $\sum_{d|n} \mu(\frac{n}{d})\ln d$
- $\sum_{d|n} \frac{\mu(d)}{d}$
- $\sum_{\substack{(t,n)=1 \\ 1\le t<n}} t$

Problem 14. Consider the function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, the so-called *Riemann zeta-function.* It converges for $s > 1$. In fact, one can extend it to an *analytic* function over the whole complex plane except $s = 1$. The famous Riemann Conjecture claims that all zeros of $\zeta(s)$ in the strip $0 \le \operatorname{Re} s \le 1$ lie on the line $\operatorname{Re} s = 1/2$. For $\zeta(s)$ show the following formal identities:

- $\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}}$
- $\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$
- $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$
- $\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$

Problem 15. Suppose that we are given infinitely many tickets, each with one natural number on it. For any $n \in \mathbb{N}$, the number of tickets on which divisors of n are written is exactly n. For example, the divisors of 6, $\{1, 2, 3, 6\}$, are written in some variation on 6 tickets, and no other ticket has these numbers written on it. Prove that any number $n \in \mathbb{N}$ is written on at least one ticket.

Problem 16. Let $f(n) : \mathbb{N} \to \mathbb{N}$ be multiplicative and strictly increasing. If $f(2) = 2$, then $f(n) = n$ for all n.

7. Problems with $[x]$ and Multiplicative Functions

Problem 17. Prove that $\sum_{k=1}^{n}\tau(k)=\sum_{k=1}^{n}\left[\frac{n}{k}\right]$ and $\sum_{k=1}^{n}\sigma(k)=\sum_{k=1}^{n}k\left[\frac{n}{k}\right]$.

Problem 18. Prove that for $(p,q)=1$: $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]=\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]$.

Problem 19. Prove that for $\forall\, n\in\mathbb{N}$: $\sum_{1\le k<n}\left\lfloor\frac{\varphi(k+1)}{k}\right\rfloor=n-1-\sum_{k=1}^{n}\left\lceil\left\{\frac{(k-1)!+1}{k}\right\}\right\rceil$.

Problem 20. Let $S(m,n)=\left\{k\in\mathbb{Z}\,\middle|\, m\,(\mathrm{mod}\,k)+n\,(\mathrm{mod}\,k)\ge k\right\}$. Find $\sum_{k\in S(m,n)}\varphi(k)$.

8. Hints and Solutions to Selected Problems

Hint 10. Define a function $a:\mathbb{N}\to\mathbb{C}$ by

$$a(n)=\begin{cases} a_0 & \text{if } n=1\\ a_r & \text{if } n=p_1p_2\cdots p_r\\ 0 & \text{if } n-\text{not sq.free.}\end{cases}$$

Solution 11. We have $\varphi(2^{n+1}-1)=2^n$. If $2^{n+1}-1=1$, then $n=0$ and $\varphi(1)=1$. Otherwise, $2^{n+1}-1=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}>1$ with all p_i's odd. From a previous formula,

$$\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r})=2^n=p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_r^{\alpha_r-1}(p_1-1)(p_2-1)\cdots(p_r-1).$$

Therefore, all $\alpha_i=1$, $2^n=(p_1-1)(p_2-1)\cdots(p_r-1)$, and all $p_i=2^{s_i}+1$ for some $s_i\ge 1$. It is easy to see that if s_i has an odd divisor >1, then $2^{s_i}+1$ will factor, making p_i non-prime. Hence $p_i=2^{2^{q_i}}+1$ for some $q_i\ge 0$.

We have reduced the problem to finding all sets of primes p_i of the above type such that

$$p_1p_2\cdots p_r+1=2^{n+1}=2(p_1-1)(p_2-1)\cdots(p_r-1).$$

Order the primes $p_1<p_2<\cdots<p_r$. Check consecutively via modulo appropriate powers of 2, that, if they exist, the smallest primes must be: $p_1=3$, $p_2=5$, $p_3=17$, $p_4=257$. However, if p_5 also exists, then $p_5=2^{2^4}+1$, which is not prime. Hence, $r\le 4$, and $n=1,2,3,4$. □

Hint 12. Modify the "table" proof for multiplicativity of the Euler function $\varphi(n)$.

Hint 13, "ln$d$". Note that $\ln d$ is *not* multiplicative. Consider the function $\wedge(n)$ defined by: $\wedge(n)=\ln p$ if n is a power of a prime p, and $\wedge(n)=0$ otherwise, and use Möbius inversion.

Solution 16. We have $f(1)=1$ and $f(2)=2$. Let $f(3)=3+m$ for $m\in\mathbb{N}\cup\{0\}$.

$$\Rightarrow\ f(6)=f(2)\cdot f(3)=6+2m\ \Rightarrow\ f(5)\le 5+2m\ \Rightarrow\ f(10)\le 10+4m$$
$$\Rightarrow\ f(9)\le 9+4m\ \Rightarrow\ f(18)\le 18+8m\ \Rightarrow\ f(15)\le 15+8m.$$

But $f(15)=f(3)\cdot f(5)\ge(3+m)\cdot(5+m)=15+8m+m^2$. Hence, $m^2\le 0$, i.e. $m=0$ and $f(3)=3$.

Now assume by induction on k that $f(s) = s$ for $s = 1, 2, ..., 2k-1$, where $k \geq 2$. Then $f(4k-2) = 2f(2k-1) = 4k-2$. Since $f(2k-1) = 2k-1$ and the function is strictly increasing, $f(s) = s$ for all $s \in [2k-1, 4k-2]$. In particular, $f(2k) = 2k$ and $f(2k+1) = 2k+1$ $(4k-2 > 2k+1)$. This completes the induction step. $\square$

Solution 17a. Set $B_{n,k} = \left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right]$ for $k = 1, 2, ..., n$. Then $B_{n,k} = \begin{cases} 1 \text{ when } k\,|n, \\ 0 \text{ when } k\not|n, \end{cases}$ and $\tau(n) = \sum_{k=1}^{n} B_{n,k}$. Summing up, we obtain

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n}\sum_{m=1}^{k} B_{m,k} = \sum_{k=1}^{n}\sum_{m=1}^{k}\left(\left[\frac{k}{m}\right] - \left[\frac{k-1}{m}\right]\right) =$$

$$= \sum_{k,m=1}^{n} \left[\frac{k}{m}\right] - \sum_{\substack{1\leq m\leq n\\ 1\leq k\leq n-1}} \left[\frac{k}{m}\right] = \sum_{\substack{1\leq m\leq n\\ k=n}} \left[\frac{k}{m}\right]. \quad \square$$

Solution 17b. From the previous problem, $kB_{n,k} = \begin{cases} k \text{ when } k\,|n, \\ 0 \text{ when } k\not|n, \end{cases}$ and $\sigma(n) = \sum_{k=1}^{n} kB_{n,k}$.

$$\Rightarrow \sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n}\sum_{m=1}^{k} mB_{m,k} = \sum_{k=1}^{n}\sum_{m=1}^{k}\left(m\left[\frac{k}{m}\right] - m\left[\frac{k-1}{m}\right]\right) =$$

$$= \sum_{k,m=1}^{n} m\left[\frac{k}{m}\right] - \sum_{\substack{1\leq m\leq n\\ 1\leq k\leq n-1}} m\left[\frac{k}{m}\right] = \sum_{\substack{1\leq m\leq n\\ k=n}} m\left[\frac{k}{m}\right]. \quad \square$$

Hint 18. Use the identity $\left[\frac{kp}{q}\right] - \left[\frac{(q-k)p}{q}\right] = p-1$ for $k = 1, 2, ..., q-1$, and show that both sums equal $\frac{(p-1)(q-1)}{2}$.

Hint 19. Both sides count the number of primes $p \leq n$.

Hint 20. The given sum equals

$$\sum_{k}\left(\left[\frac{m+n}{k}\right] - \left[\frac{m}{k}\right] - \left[\frac{n}{k}\right]\right)\varphi(k) = \Sigma(m+n) - \Sigma(m) - \Sigma(n),$$

where $\Sigma(l) = \sum_{k}\left[\frac{l}{k}\right]\varphi(k)$. Further,

$$\Sigma(l) = \sum_{1\leq q\leq l}\sum_{d|q}\varphi(d) = \binom{l+1}{2}.$$