BAMO ADVANCED PREPARATION SECTION

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1. First Hour

1.1. Simpler Problems.

Problem 1.1.1. Prove that $m(m+1)$ cannot be a power (greater than 1) of an integer for any natural m.

Problem 1.1.2. Let f be a polynomial with integer coefficients and roots $\alpha_1, \ldots, \alpha_n$. Let $M = \max_i |\alpha_i|$. Prove that if for some x_0, such that $|x_0| > M + 1$, $f(x_0)$ is a prime number, then f is a prime polynomial (cannot be factored into polynomials with integer coefficients and of smaller degree)

Problem 1.1.3. Can all vertices of an equilateral triangle have integer coordinates?

Problem 1.1.4. Proof that in any triangle ABC, angle bisector AE lies between meridian AM and altitude AH.

1.2. Harder Problems.

Problem 1.2.1. Cauchy Theorem: Let $f(x) = x^n - b_1x^{n-1} - \ldots - b_n$, where b_i are non-negative and at least one of them is positive. Prove that f has a single (counting the multiplicity) positive root. Moreover, the absolute values of all other roots are less than it.

Solution. Consider the function

$$F(x) = -\frac{f(x)}{x^n} = \frac{b_1}{x} + \ldots + \frac{b_n}{x^n} - 1$$

If $x \neq 0$, then $F(x) = 0$ is equivalent to $f(x) = 0$. Furthermore, for $x \in (0, +\infty)$ $F(x)$ is strictly decreasing from $+\infty$ to -1. Therefore, $F(x)$ has a single positive root p. This root has multiplicity of 1 because the derivative of F at p is non-zero (negative) and hence the derivative of f at p is non-zero.

It remains to show that all other roots' absolute value is less than p. Assume the contrary, that there exists a root x_0 such that $q = |x_0| > p$. Because F is strictly decreasing on positive reals, $F(q) < F(p) = 0$, i.e. $f(q) > 0$. On the other hand,

$$\begin{aligned} q^n &= |x_0|^n \\ &= |x_0^n| \\ &= |b_1x_0^{n-1} + \ldots + b_n| \\ &\leq |b_1x_0^{n-1}| + \ldots + |b_n| \\ &= b_1|x_0^{n-1}| + \ldots + b_n \\ &= b_1q^{n-1} + \ldots + b_n \end{aligned}$$

which is equivalent to $f(q) \leq 0$. Thus, we find a contradiction and complete the proof $\square$

Problem 1.2.2. Let $A_1A_2 \ldots A_{2m}$ be a convex $2m$-gon. Point P is taken inside it such that it does not lie on any diagonal. Prove that P lies in an even number of triangles with vertices in points $A_1, A_2, \ldots, A_{2m}$.

Solution Sketch. Draw all the diagonals of the $2m$-gon and consider all the small regions outlined by the diagonals. Call the regions that share a side neighbors. If PQ is the shared side of two neighbors, note that any triangle with vertices in $A_1, A_2, \dots, A_{2m}$ that does not have PQ as its side either completely covers both of the regions or does not intersect both of the regions. Using this show that as we go from one neighbor to the other, the number of triangles that cover the region does not change its parity. Finally, find a region that is covered by an even number of triangles and note that one can get to any other internal point of the $2m$-gon from this region by going from a neighboring region to a neighboring region. □

Problem 1.2.3. Is there a closed broken line with odd number of segments of equal length each vertex of which has integer coordinates?

Soluiton Sketch. No. Assume there is such a broken line $A_1A_2\dots A_nA_1$. Let c be the length of each of its segments. Let a_i and b_i be the coordinates of the projection of vector A_iA_{i+1} on the horizontal and vertical axes, respectively. All a_i and b_i are integers. Then, $c^2 = a_i^2 + b_i^2$. Note that c^2 is an integer and can only have have remainders 0, 1, or 2 when divided by 4 (because a square of an integer can only have remainders of 0 or 1 when divided by 4). If c^2 is divisible by 4, all a_i and b_i are even. Then, we can find a smaller broken line with the same properties. Continue after we find a line such that c^2 has remainder of 1 or 2. Because the line is closed, $a_1 + \dots + a_n = b_1 + \dots + b_n$. Consider the two cases and find simple parity-based contradictions. □

2. Second Hour

2.1. **Simpler Problems.**

Problem 2.1.1. Natural numbers a and b are coprime. Prove that the greatest common divisor of $a+b$ and a^2+b^2 is either 1 or 2.

Solution Hint. Consider $(a+b)^2 - (a^2+b^2) = 2ab$. If prime p divides both $a+b$ and a^2+b^2, it has to divide $2ab$. □

Problem 2.1.2. There are 3 non-colinear pucks on the ground. A hockey player always hits a puck such that it goes between the other two and does not stop of the line formed by them. Can the player after 25 hits return each puck to its original position.

Soluiton Hint. No, because each time the orientation of the pucks changes. □

Problem 2.1.3. Given a rectangle $ABCD$ and 4 circles C_1, C_2, C_3, C_4 with centers at A, B, C, D and radii r_1, r_2, r_3, r_4. Moreover, $r_1 + r_3 = r_2 + r_4 < d$, where d is the diagonal of the rectangle. Two pairs of external tangents are drawn to circles C_1, C_3 and C_2, C_4. Prove that one can inscribe a circle into the quadrilateral formed by these 4 lines.

Solution Hint. Consider the circle at the center of the rectangle with radius $(r_1 + r + 3)/2$ □

Problem 2.1.4. Are there natural numbers x and y such that $x^2 + y$ and $x + y^2$ are squares of natural numbers.

Solution. WLOG assume that $y \leq x$. Then

$$x^2 < x^2 + y \leq x^2 + x < (x+1)^2$$

Therefore, $x^2 + y$ cannot be a square of an integer □

2.2. **Harder Problems.**

Problem 2.2.1. In triangle ABC sides AC and BC are not equal. Prove that angle bisector CE of angle $\angle ACB$ equally bisects the angle between the meridian CM and altitude CH if and only if, angle $\angle ACB$ is 90^o

Solution. Only if part: Let D be the intersection point of the circumscribed circle of ABC and CE. Then, MD is perpendicular to AB and hence parallel to CH. Thus, $\angle MCD = \angle HCD = \angle MDC$. Thus, $MD = MC$ which is only possible if M is the center of the circle. The "if" part is straightforward application of this argument in the opposite direction. $\square$

Problem 2.2.2. Prove that for any three infinite sequences of natural numbers $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, there exist two indices p and q such that $a_p \geq a_q$, $b_p \geq b_q$, and $c_p \geq c_q$.

Solution Sketch. Prove that there exist and consider an infinite non-decreasing subsequence of $\{a_i\}$. Using just the indices from this subsequence find an infinite non-decreasing subsequence of $\{b_i\}$. Using just the indices from this subsequence of $\{b_i\}$ find an infinite non-decreasing subsequence of $\{c_i\}$. Any to indices from this subsequence of $\{c_i\}$ can be p and q. $\square$

Problem 2.2.3. Nine out of 100 cells of a 10x10 board are occupied by a Marcian bacteria. Each day it spreads to all cells that have at least two neighboring cell (that sharing an edge) already occupied. Prove that it will never occupy all the cells. What if initially 10 cells are occupied?

Solution Sketch. Notice that the total perimeter of all occupied region does not increase. In the beginning it is at most $9 \times 4 = 36$. If the bacteria were to cover all the cells, the perimeter would have been 40. $\square$

Problem 2.2.4. Suicidal King: On a $1000x1000$ chess board, there is a black king and 499 white rocks. Prove that the king can always move in such a way as to be killed by a rock, independently of the initial configuration.

Solution Sketch. The king can come to the bottom left corner and then go up to the top right corner, unless it can win by moving back. Then, before the second step of the king, all the rocks should be above the 3rd horizontal and to the right of the 3rd vertical line. Likewise, before the last move by the king, all the rocks have to be below the 997th horizontal and to the left of the 997th vertical line. Between these two time points. The king has to make 997 moves, but each rock has to make at least two moves (change its horizontal and vertical position) thus the total moves by the rocks have to be at least $2 \times 499 = 998$. $\square$

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