Berkeley Math Circle
Monthly Contest 8 – Solutions

1. Determine whether there exists a natural number having exactly 10 divisors (including itself and 1), each ending in a different digit.

   Solution. The answer is no.

   Suppose that such a number n exists. Since n has a divisor ending in 5, n is divisible by 5. Then for each divisor d of n not divisible by 5, $5d$ also divides n. This contradicts the assumption that n has eight divisors ending in 1, 2, 3, 4, 6, 7, 8, or 9 and only two ending in 5 or 0.

2. Two bikers, Bill and Sal, simultaneously set off from one end of a straight road. Neither biker moves at a constant rate, but each continues biking until he reaches one end of the road, at which he instantaneously turns around. When they meet at the opposite end from where they started, Bill has traveled the length of the road eleven times and Sal seven times. Find the number of times the bikers passed each other moving in opposite directions.

   Solution. Define a *pass* (P) to be an time when Bill and Sal pass one another moving in opposite directions and a *turn* (T) to be a time when one of the bikers turns around. If both bikers turn around simultaneously, we may alter their speeds slightly, causing one turn to happen before the other, without affecting the number of passes. We distinguish three states:

   A. The bikers are moving in the same direction.
   B. The bikers are moving toward each other.
   C. The bikers are moving away from one another.

   We notice that the bikers can only change state by turning or passing. Moreover, there are only three possible state changes:

   - From state A, one biker reaches the end and turns around, creating state B.
   - From state B, the bikers pass one another, creating state C.
   - From state C, one biker reaches the end and turns around, creating state A.

   Since there are no other possibilities and we begin at state A, the states must follow the sequence ABCABC$\cdots$ABCA and the turns and passes TPTTPT$\cdots$TPT. We are given that Bill makes ten turns and Sal six, or 16 in all. Consequently there are eight passes.

3. A position of the hands of a (12-hour, analog) clock is called *valid* if it occurs in the course of a day. For example, the position with both hands on the 12 is valid; the position with both hands on the 6 is not. A position of the hands is called *bivalid* if it is valid and, in addition, the position formed by interchanging the hour and minute hands is valid. Find the number of bivalid positions.

   Solution. Let h and m denote the respective angles of the hour and minute hands, measured clockwise in degrees from 12 o'clock ($0 \leq h, m < 360$). Since the minute hand moves twelve times as fast as the hour hand, we have

   $$m = 12h - 360a, \qquad (1)$$

   with a an integer, for any valid time. Conversely, it is clear that any ordered pair (h, m) satisfying (1) represents the valid time of $h/30$ hours after 12 o'clock.

   The condition that a time be valid after switching the hour and minute hands is, of course,

   $$h = 12m - 360b, \qquad (2)$$

   with b an integer. Thus the problem of finding bivalid times is reduced to finding pairs (h, m) satisfying (1) and (2). Note that if h is known, m is uniquely determined from (1) and the condition $0 \leq m < 360$. Note also that the truth of (2) is not affected by increasing or decreasing m by 360, as this simply increases or decreases b by 12, giving a new integer value b'. Consequently we substitute $12h$ for m in (2), getting

$$\begin{aligned} h &= 144h - 360b' \\ 143h &= 360b' \\ h &= \frac{360b'}{143}. \end{aligned}$$

This equation has 143 solutions in the range $0 \leq h < 360$, each of which gives rise to one solution of (1) and (2). We conclude that there are 143 bivalid positions.

4. Let ABC be a triangle, and let M and N be the respective midpoints of AB and AC. Suppose that

$$\frac{CM}{AC} = \frac{\sqrt{3}}{2}.$$

Prove that

$$\frac{BN}{AB} = \frac{\sqrt{3}}{2}.$$

Solution. Let L be the midpoint of BC. Let $BC = 2a$, $AC = 2b$, and $AB = 2c$. Applying the parallelogram law to $CLMN$ gives

$$CM^2 + c^2 = a^2 + b^2 + a^2 + b^2$$

or

$$CM^2 = 2a^2 + 2b^2 - c^2.$$

Squaring both sides of the given equation and substituting yields

$$\frac{2a^2 + 2b^2 - c^2}{4b^2} = \frac{3}{4}$$

which simplifies to

$$b^2 + c^2 = 2a^2.$$

This equation is equivalent to $CM/AC = \sqrt{3}/2$. Because it is symmetric in b and c, we conclude that it is also equivalent to $BN/AB = \sqrt{3}/2$.

5. Let M_n be the number of integers N such that

   (a) $0 \leq N < 10^n$;

   (b) N is divisible by 4;

   (c) The sum of the digits of N is also divisible by 4.

Prove that $M_n \neq 10^n/16$ for all positive integers n.

Solution: Since $10^n/16$ is not an integer for $n = 1, 2, 3$, we may assume that $n \geq 4$. Let

$$\tilde{M_n} = M_n - \frac{10^n}{16}.$$

We note that numbers whose hundreds digit is between 2 and 9 inclusive make no total contribution to $\tilde{M_n}$ for $n \geq 3$. This is because, given the remaining digits, divisibility by 4 (which depends only on the tens and units digits) occurs one-fourth of the time, and exactly two of the eight choices for the hundreds digit brings the total of the digits to a multiple of 4. Similarly, numbers with hundreds digit 0 or 1 but thousands digit at least 2 make no contribution to $\tilde{M_n}$. (We require the hundreds digit to be less than 2 to avoid double counting.) Continuing in this way, we find that only the $25 \cdot 2^n$ numbers with all but the last two digits equal to 0 or 1 can make $\tilde{M_n}$ nonzero. Let L_n be the number of such numbers satisfying conditions (a), (b), and (c). We thus have

$$\tilde{M_n} = L_n - \frac{25 \cdot 2^n}{16}.$$

Now let $f(i,k)$ be the number of binary strings of length k whose sum is congruent to $i \mod k$. Since there are respectively $9, 4, 6, 6$ multiples of 4 less than 100 with digit sums $\equiv 0, 1, 2, 3 \mod 4$, we have

$$L_n = 9f(0, n-2) + 4f(3, n-2) + 6f(2, n-2) + 6f(1, n-2).$$

If we let $\tilde{f}(i,k) = f(i,k) - 2^n/4$, we can subtract $25 \cdot 2^n/16$ from both sides of the above equation to get

$$\tilde{M}_n = 9\tilde{f}(0, n-2) + 4\tilde{f}(3, n-2) + 6\tilde{f}(2, n-2) + 6\tilde{f}(1, n-2). \tag{3}$$

On the other hand, dividing a binary k-string with sum i into a $k-1$-string and a single bit and noting that the sum of the $k-1$-string is either i or $i-1$, we get the recurrence

$$f(i,k) = f(i, k-1) + f(i-1, k-1)$$

which is equivalent to

$$\tilde{f}(i,k) = \tilde{f}(i, k-1) + \tilde{f}(i-1, k-1). \tag{4}$$

Together with the boundary conditions

$$\begin{aligned} f(0,0) &= 1, & f(1,0) = f(2,0) = f(3,0) &= 0, \\ \tilde{f}(0,0) &= \frac{3}{4}, & \tilde{f}(1,0) = \tilde{f}(2,0) = \tilde{f}(3,0) &= -\frac{1}{4}, \end{aligned} \tag{5}$$

this enables us to calculate values of $\tilde{f}(i,k)$ and therefore of $\tilde{M}_n$. It is trivial to prove by induction that for $k \geq 1$,

$$\tilde{f}(i,k) = -\tilde{f}(i+2, k),$$

from which it follows after a short computation using (4) that

$$\tilde{f}(i, k+4) = -4\tilde{f}(i,k), \quad k \geq 1.$$

Now we substitute this equation into (3), yielding

$$\tilde{M}_{n+4} = -4\tilde{M}_n, \quad n \geq 3.$$

This implies that $\tilde{M}_{n+4}$ is nonzero if $\tilde{M}_n$ is. We have already noted that $\tilde{M}_3$ is nonzero because it is not an integer. Computing $\tilde{M}_4 = 2$, $\tilde{M}_5 = -1$, and $\tilde{M}_6 = -6$ by hand using (5), (4), and (3) completes the proof that $\tilde{M}_n$ is nonzero for all $n \geq 1$.