Berkeley Math Circle
Monthly Contest 7
Due March 31, 2009

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 7
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

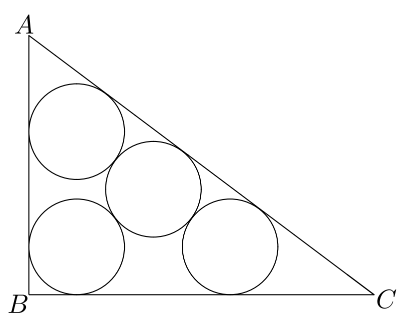
Enjoy solving these problems and good luck!

Problems

1. In the sequence
$$7749283618162418688 6128\ldots,$$
all of the digits except the first two are obtained by writing down the products of pairs of consecutive digits. Prove that infinitely many 6s appear in the sequence.

2. Let k be a positive rational number. Prove that there exist positive integers a, b, c satisfying the equations
$$a^2 + b^2 = c^2 \tag{1}$$
$$\frac{a+c}{b} = k. \tag{2}$$

3. Four congruent circles are tangent to each other and to the sides of a triangle as shown.



   (a) Prove that $\angle ABC = 90°$.

   (b) If $AB = 3$ and $BC = 4$, find the radius of the circles.

4. Find all pairs (a, b) of positive integers such that
$$1 + 5^a = 6^b.$$

5. The *tower function of twos*, $T(n)$, is defined by $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Prove that $T(n) - T(n-1)$ is divisible by $n!$ for $n \geq 2$.