Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Let p, q, and r be distinct primes. Prove that $p+q+r+pqr$ is composite.

   Solution. Note that at most one of p, q, and r can equal 2. If $p=2$, then q and r are both odd, but pqr is even. Therefore their sum is even. Similarly, if q or r is 2, then $p+q+r+pqr$ is even. Finally, if p, q, and r are all odd, then so is pqr, and $p+q+r+pqr$ is even. Thus $p+q+r+pqr$ is always divisible by 2, and since it is obviously greater than 2, it must be composite.

2. The sequence
$$5,\ 9,\ 49,\ 2209, \ldots$$
is defined by $a_1 = 5$ and $a_n = a_1 a_2 \cdots a_{n-1} + 4$ for $n > 1$. Prove that a_n is a perfect square for $n \geq 2$.

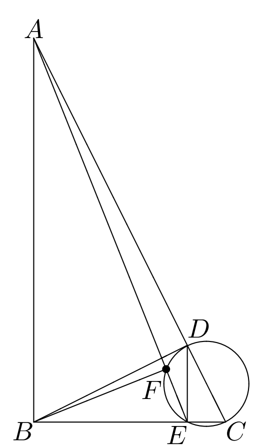
   Solution. This is clear for $n = 2$. We use the relation
$$\begin{aligned} a_{n-1} &= a_1 a_2 \cdots a_{n-2} + 4 \\ a_1 a_2 \cdots a_{n-2} &= a_{n-1} - 4 \end{aligned}$$
for $n \geq 3$ to transform a_n:
$$\begin{aligned} a_n &= a_1 a_2 \cdots a_{n-2} a_{n-1} + 4 \\ &= (a_{n-1} - 4)\, a_{n-1} + 4 \\ &= a_{n-1}^2 - 4a_{n-1} + 4 \\ &= (a_{n-1} - 2)^2 . \end{aligned}$$
This is clearly the square of an integer.

3. The integers from 1 to 13 are arranged around several rings such that every number appears once and every ring contains at least one two-digit number. Prove that there exist three one-digit numbers adjacent to one another on one ring.

   Solution. For any $n \in S = 1, 2, \ldots, 13$, define $f(n)$ to be the number immediately clockwise of n on the same ring, where $f(n) = n$ if n lies on a one-element ring. Notice that f is a bijective function, since every number is immediately clockwise of exactly one number. Notice that there are four numbers $n \in S$ such that n has two digits, four numbers n such that $f(n)$ has two digits, and four numbers n such that $f(f(n))$ has two digits. This leaves at least $13 - 4 - 4 - 4 = 1$ number n such that n, $f(n)$, and $f(f(n))$ are all one-digit numbers. No two of them can be equal, or else the ring containing them would have no two-digit numbers, so n, $f(n)$, and $f(f(n))$ are the desired adjacent one-digit numbers.

4. Let ABC be a triangle with $\angle ABC = 90°$. Points D and E on AC and BC respectively satisfy $BD \perp AC$ and $DE \perp BC$. The circumcircle of $\triangle CDE$ intersects AE at two points, E and F. Prove that $BF \perp AE$.

   Solution: By Power of a Point,
$$AF \cdot AE = AD \cdot AC;$$
because $\triangle ABC$ is right,
$$AD \cdot AC = AB^2.$$
Combining,
$$AF \cdot AE = AB^2.$$
On the other hand, if F' is the foot of the altitude from B to AE, then
$$AF' \cdot AE = AB^2.$$
Consequently $AF' = AF$ and $F' = F$.



Remark. The right angle at D is a red herring. The proof is valid for any segment DE parallel to leg AB.

5. Let $a_1,\ a_2, \ldots,\ a_n$ be distinct integers. Prove that there do not exist two nonconstant integer-coefficient polynomials p and q such that

$$(x - a_1)(x - a_2)\cdots(x - a_n) - 1 = p(x)q(x) \tag{1}$$

for all x.

Solution. Assume for the sake of contradiction that p and q exist. If we substitute $x = a_i$ for $i = 1, \ldots, n$, the left side of (1) becomes -1. Since $p(a_i)$ and $q(a_i)$ are both integers, we either have

$$p(a_i) = 1, \quad q(a_i) = -1$$

or

$$p(a_i) = -1, \quad q(a_i) = 1.$$

In either case,

$$(p + q)(a_i) = 0.$$

Thus $p + q$ is a polynomial with n distinct roots $a_1, a_2, \ldots, a_n$. Such a polynomial must have degree at least n—unless it is the zero polynomial.

Case 1. $p + q$ has degree $\geq n$. Then one of p and q has degree $\geq n$, and the other, of course, has degree ≥ 1. It follows that pq has degree $\geq n + 1$, a contradiction since the left side of (1) has degree n.

Case 2. $p(x) + q(x) = 0$ for all x. Substituting $q = -p$ into (1),

$$(x - a_1)(x - a_2)\cdots(x - a_n) - 1 = -p(x)^2.$$

Now substitute an integer for x that is so huge that each of the factors $(x - a_i)$ is greater than 1. This makes the left side strictly positive. Since the right side is negative or zero, we have a contradiction.