Berkeley Math Circle
Monthly Contest 2 – Solutions
Due November 4, 2008

1. Each square of a 100×100 grid is colored black or white so that there is at least one square of each color. Prove that there is a point with is a vertex of exactly one black square.

   Solution. Locate the uppermost row that has at least one black square. Then, within that row, find the leftmost black square. By construction, all squares above and/or to the left of that square are white. Therefore, the upper left corner of that square will solve the problem.

2. Let a and b be nonzero real numbers. Prove that at least one of the following inequalities is true:

$$\left|\frac{a+\sqrt{a^2+2b^2}}{2b}\right| < 1 \qquad (1)$$

$$\left|\frac{a-\sqrt{a^2+2b^2}}{2b}\right| < 1 \qquad (2)$$

   Solution. Assume for the sake of contradiction that both inequalities are false, that is,

$$1 \le \left|\frac{a+\sqrt{a^2+2b^2}}{2b}\right| \text{ and } 1 \le \left|\frac{a-\sqrt{a^2+2b^2}}{2b}\right|.$$

   Multiplying these inequalities together yields

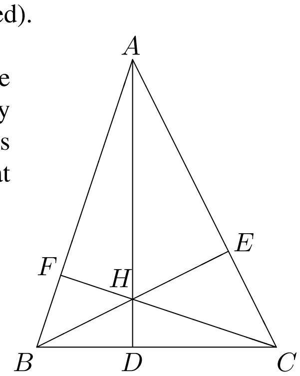
$$1 \le \left|\frac{a+\sqrt{a^2+2b^2}}{2b} \cdot \frac{a-\sqrt{a^2+2b^2}}{2b}\right| = \left|\frac{a^2-(a^2+2b^2)}{4b^2}\right| = \left|\frac{-2b^2}{4b^2}\right| = \frac{1}{2},$$

   a contradiction.

3. In acute triangle ABC, the three altitudes meet at H. Given that $AH = BC$, calculate at least one of the angles of $\triangle ABC$.

   Answer: $\angle A = 45°$ (angles B and C cannot be determined).

   Solution: In the diagram, $\angle AFH = \angle CFB$ (both are right angles) and $\angle FAH = \angle FCB$ (both are complementary to $\angle ABC$). We get $\triangle AFH \cong \triangle CFB$ by AAS. It follows that $AF = FC$, so $\triangle AFC$ is right isosceles. We find that $\angle BAC = 45°$.



4. Let x be an integer greater than 2. Prove that the binary representation of $x^2 - 1$ has at least three consecutive identical digits (000 or 111).

Solution. We consider three cases:

- *x is odd.* Then $x+1$ and $x-1$ are consecutive even integers, and thus one of them must be divisible by 4, the other only by 2. So $x^2-1=(x+1)(x-1)$ is divisible by 8; since $x \geq 3$, x^2-1 has four or more digits, the last three of which are 0's.
- *x is even but $\frac{x}{2}$ is odd.* By the same reasoning, $\left(\frac{x}{2}\right)^2-1=8k$ for some integer k, so

$$x^2-1=4\left(\frac{x}{2}\right)^2-1=4(8k+1)-1=32k+3.$$

  Since $x \geq 6$, this number has six or more digits, the last five of which are 00011. Thus we also get three consecutive zeros in this case.
- *x and $\frac{x}{2}$ are both even,* that is, x is divisible by 4. Then x^2 is divisible by 16. Since $x \geq 4$, x^2-1 has at least four digits, the last four of which are 1's.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f\left(x(1+y)\right)=f(x)\left(1+f(y)\right) \tag{3}$$

for all $x,\ y \in \mathbb{R}$.

Solution. The answers are $f(x)=0$ and $f(x)=x$. It is easily checked that both of these satisfy the equation.

If $f(x)=c$, a constant, for all x, we must have

$$\begin{aligned} c &= c(1+c) \\ c &= c+c^2 \\ 0 &= c^2 \\ 0 &= c, \end{aligned}$$

and the solution $f(x)=0$ has already been noted. Thus we may assume that f is not a constant.

Plugging $y=-1$ into (3) gives

$$\begin{aligned} f(x(1-1)) &= f(x)(1+f(-1)) \\ f(0) &= f(x)(1+f(-1)) \end{aligned}$$

If $f(-1) \neq -1$, we have $f(x)=\frac{f(0)}{1+f(-1)}$, a constant. Thus we may assume that $f(-1)=-1$. It follows that $f(0)=0$. Next we try to find $u := f(1)$. Plugging $x=1$ and $y=-2$ into (3) gives

$$\begin{aligned} f(1(1-2)) &= f(1)(1+f(-2)) \\ f(-1) &= f(1)(1+f(-2)) \\ -1 &= u(1+f(-2)). \end{aligned} \tag{4}$$

On the other hand, with $x=-1$ and $y=1$ in (3), we get

$$\begin{aligned} f(-(1+1)) &= f(-1)(1+f(1)) \\ f(-2) &= -(1+u), \end{aligned}$$

and plugging this into (4),

$$\begin{aligned} -1 &= u(1-(1-u)) \\ &= u(-u) \\ 1 &= u^2, \end{aligned}$$

so $u = 1$ or $u = -1$. If $u = -1$, using $y = 1$ in (3) yields

$$\begin{aligned} f(x(1+1)) &= f(x)(1+f(1)) \\ f(2x) &= f(x)(1-1) = 0, \end{aligned}$$

and f becomes a constant. Thus we must have $f(1) = 1$. Using $x = 1$ in (3),

$$f(1+y) = f(1)(1+f(y)) = 1 + f(y).$$

We can use this identity to transform (3):

$$\begin{aligned} f(x(1+y)) &= f(x)(1+f(y)) \\ &= f(x)f(1+y), \end{aligned}$$

and letting $z = 1 + y$, we get

$$f(xz) = f(x)f(z) \tag{5}$$

for all x and z. Now we transform (3) in yet another way:

$$\begin{aligned} f(x(1+y)) &= f(x)(1+f(y)) \\ f(x+xy) &= f(x) + f(x)f(y) \\ &= f(x) + f(xy); \end{aligned}$$

letting $z = xy$,

$$f(x+z) = f(x) + f(z) \tag{6}$$

whenever $x \neq 0$. Since this equation is trivially true when $x = 0$, we can use it for all x and z. Repeatedly applying it to the known values $f(1) = 1$, $f(-1) = -1$, we find $f(x) = x$ whenever x is an integer. Then, if $\frac{m}{n}$ is any rational number, plugging $x = \frac{m}{n}$, $z = n$ into (5) shows that $f(x) = x$ for rational x as well. Setting $x = y$ in (5) gives

$$f(x^2) = (f(x))^2,$$

so if x is nonnegative, so is $f(x)$. By (6), if $x \geq y$, then $f(x) \geq f(y)$. Suppose that $f(a) > a$ for some irrational a. Then there is a rational p with $f(a) > p > a$, and

$$f(a) \leq f(p) = p < f(a),$$

a contradiction. Similarly if $f(a) < a$, we get a contradiction. Thus $f(x) = x$ for all x, the second of the two solutions to the functional equation.