Berkeley Math Circle
Monthly Contest 2 – Solutions
Due November 4, 2008

1. Find all positive integers p such that p, $p+4$, and $p+8$ are all prime.

   Solution. If $p = 3$, then $p+4 = 7$ and $p+8 = 11$, both prime. If $p \neq 3$, then p is not a multiple of 3 and is therefore of one of the forms $3k+1$, $3k+2$ $(k \geq 0)$ If $p = 3k+1$, then $p+8 = 3k+9 = 3(k+3)$, which is not prime since $k+3 > 1$. If $p = 3k+2$, then $p+4 = 3k+6 = 3(k+2)$, which is not prime since $k+2 > 1$. Thus $p = 3$ is the only solution with all three numbers prime.

2. Each vertex of a regular heptagon is colored either red or blue. Prove that there is an isosceles triangle with all its vertices the same color.

   Solution. Denote the vertices of the heptagon by $ABCDEFG$. Since an alternating arrangement cannot be continued all the way around the heptagon, two adjacent vertices must be the same color, say A and B. If any of C, E, G shares this color, we are done since triangles ABC, ABE, and ABG are all isosceles. On the other hand, if C, E, and G are all of the opposite color, we are also done because triangle CEG is isosceles. Thus in all cases we can find an isosceles triangle.

3. Let a, b, and c be positive real numbers satisfying $a^b > b^a$ and $b^c > c^b$. Does it follow that $a^c > c^a$?

   Solution. Yes. We have

   $$(a^c)^b = \left(a^b\right)^c > (b^a)^c = (b^c)^a > \left(c^b\right)^a = (c^a)^b\,;$$

   the desired inequality follows by taking the bth root.

4. Let n be a positive integer and let S be the set $\{1, 2, \ldots, n\}$. Define a function $f : S \to S$ by

   $$f(x) = \begin{cases} 2x & \text{if } 2x \leq n, \\ 2n - 2x + 1 & \text{otherwise.} \end{cases}$$

   Define $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, and so on. If m is a positive integer satisfying $f^m(1) = 1$, prove that $f^m(k) = k$ for all $k \in S$.

   Solution. First note that

   $$f(x) \equiv \pm 2x \mod 2n+1.$$

   It follows that

   $$f^p(x) \equiv \pm 2^p x \mod 2n+1.$$

   Thus if $f^m(1) = 1$, $2^m \equiv \pm 1$ and so, for any $k \in S$,

   $$f^m(k) \equiv \pm 2^m k \equiv \pm k \mod 2n+1,$$

   that is, $f^m(k) \pm k = j(2n+1)$ for some integer j and some choice of the sign. Since

   $$0 < 1 + 1 \leq f^m(k) + k \leq n + n < 2n+1,$$

   the plus sign is invalid. Thus the minus sign holds, and since

   $$-(2n+1) < 1 - n \leq f^m(k) - k \leq n - 1 < 2n+1,$$

   we get $j = 0$, i.e. $f^m(k) = k$.

5. *This problem was invalid on the contest. Correct formulation as of December 9.* Let ω_1, ω_2, and ω_3 be three circles passing through the origin O of the coordinate plane but not tangent to each other or to either axis. Denote by $(x_i, 0)$ and $(0, y_i)$, $1 \leq i \leq 3$, the respective intersections (besides O) of circle ω_i with the x and y axes. Prove that ω_1, ω_2, and ω_3 have a common point $P \neq O$ if and only if the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear.

Solution. First of all, note that if a circle passes through $(0, 0)$, $(x_i, 0)$, and $(0, y_i)$, its center must be $(\frac{x_i}{2}, \frac{y_i}{2})$, the midpoint of the side opposite the right angle at O. Also note that the three points $(\frac{x_i}{2}, \frac{y_i}{2})$ are related to (x_i, y_i) by a dilation about O; thus the former three will be collinear if and only if the latter three are. It suffices to prove that the circles have a second common point if and only if their centers are collinear.

If the centers are collinear, all three circles are symmetric about the line of centers. Thus the reflection of O about this line is a second common point of the three circles. Conversely, assume that the circles have two common points, O and P. Then all three centers lie on the perpendicular bisector of OP.