Berkeley Math Circle
Monthly Contest 1 – Solutions

1. If x and y are integers such that
$$x^2y^2 = x^2 + y^2,$$
prove that $x = y = 0$.

Solution. We move all the terms to the left side of the equation and add 1.

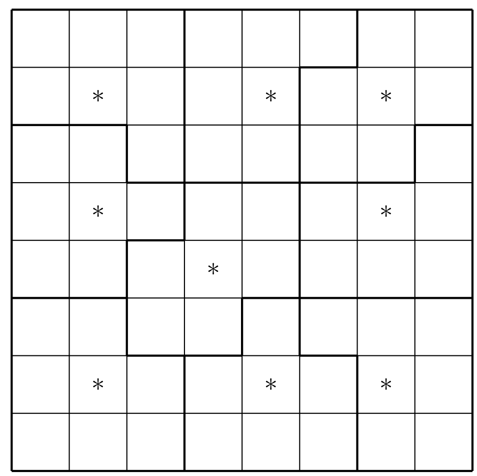
$$x^2y^2 - x^2 - y^2 + 1 = 1$$
$$\left(x^2 - 1\right)\left(y^2 - 1\right) = 1$$

Since the factors are integers, they must be both 1 or both -1. If both are 1, we get $x^2 = 2$ which is impossible for integer x. If both are -1, we get $x^2 = 0$ and $y^2 = 0$, so both x and y are zero.

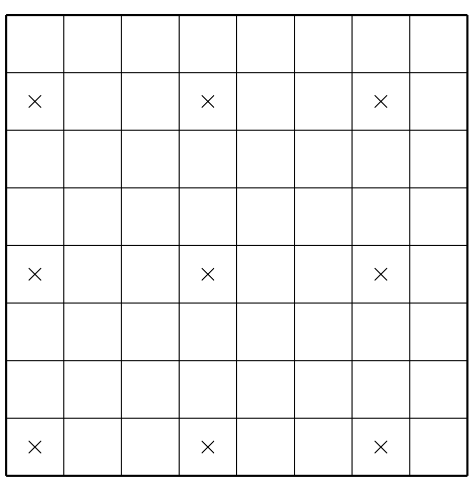
2. The country of Squareland is shaped like a square and is divided into 64 congruent square cities. We want to divide Squareland into states and assign to each state a capital city so that the following rules are satisfied:

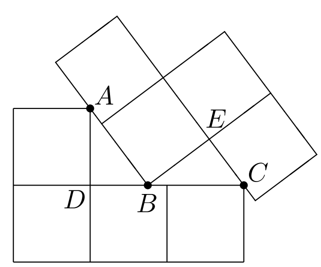
   (a) Every city lies entirely within one state.
   (b) Given any two states, the numbers of cities in them differ by at most 1.
   (c) Any city in a state shares at least one corner with the state's capital.

   What is the smallest possible number of states?



Solution. In the diagram below, no city shares a corner with any two of the cities marked X. Therefore the nine X's are in nine different states. The diagram at right shows that nine states are also sufficient ($*$ denotes capital).





3. Eight unit squares are glued together to make two groups of four, and the groups are pressed together so as to meet in three points A, B, C as shown in the diagram. Find the distance AB.

Solution. Since $AD = 1 = BE$, $\angle ADB = 90 = \angle BEC$, and $\angle ABD = 90 - \angle EBC = \angle BCE$, triangles ADB and BEC are congruent. Let $AB = BC = x$. Notice that $DB = 2 - BC = 2 - x$. We apply the Pythagorean theorem to triangle ABD:

$$\begin{aligned} 1+(2-x)^2 &= x^2 \\ 1+4-4x+x^2 &= x^2 \\ 5 &= 4x \\ \frac{5}{4} &= x \end{aligned}$$

Thus $AB = \frac{5}{4} = 1\frac{1}{4} = 1.25$ (all three answers are correct).

4. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$a(a-1) + b(b-1) + c(c-1) \geq 0.$$

Solution. At least two of a, b, c are either not less than 1 or not greater than 1. Assume that a and b are on the same side of 1. Next, transform the inequality as follows:

$$\begin{aligned} a(a-1)+b(b-1)+c(c-1) &\overset{?}{\geq} 0 \\ a(a-1)+b(b-1)+c^2\left(1-\frac{1}{c}\right) &\overset{?}{\geq} 0 \\ a(a-1)+b(b-1)+c^2(1-ab) &\overset{?}{\geq} 0 \\ a(a-1)+b(b-1)-c^2(a-1)-c^2(ab-a) &\overset{?}{\geq} 0 \\ (a-c^2)(a-1)+(b-c^2a)(b-1) &\overset{?}{\geq} 0 \\ \left(a-\frac{1}{a^2b^2}\right)(a-1)+\left(b-\frac{1}{a^2b}\right)(b-1) &\overset{?}{\geq} 0 \end{aligned}$$

Using the hypotheses concerning a and b, it is not hard to see that the four factors in parentheses are all nonnegative or all nonpositive, and therefore the left side is nonnegative.

5. The positive integers from 1 to 100 are written, in some order, in a 10×10 square. In each row, the five smallest numbers are crossed out. In each column, the five largest numbers (including those that have already been crossed out) are circled. Prove that at least 25 numbers will be circled but not crossed out.

Solution. We generalize to the following:

> Let $m \geq p$, $n \geq q$ be integers. Each square of an m-row, n-column grid is filled with a different positive integer. Then at least pq numbers are among the p largest numbers in their columns and the q largest numbers in their rows.

We prove this latter statement by induction on $m + n$. The base case, when $m = 1$ or $n = 1$, is trivial. Given an $m \times n$ grid with $m \geq 2$ and $n \geq 2$, let L be the largest number that is one of the largest in its row or its column, but not both. Assume without loss of generality that L is among the largest in its column and not among the largest in its row, so the q largest numbers in L's row are all larger than L and all among the largest in their columns. Temporarily remove this row from the matrix. By the induction hypothesis, at least $(p-1)q$ numbers are now among the q largest numbers in their rows and the $p-1$ largest numbers in their columns. These are also among the p largest numbers in their columns in the original matrix. Adding the q largest numbers in the deleted row, we find $(p-1)q + q = pq$ numbers satisfying the original condition. The problem follows by taking $m = n = 10$ and $p = q = 5$.