Berkeley Math Circle
Monthly Contest 1
Due October 7, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 1
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. If x and y are integers such that
$$x^2 + y^2 = x^2y^2,$$
prove that $x = y = 0$.

2. The country of Squareland is shaped like a square and is divided into 64 congruent square cities. We want to divide Squareland into states and assign to each state a capital city so that the following rules are satisfied:

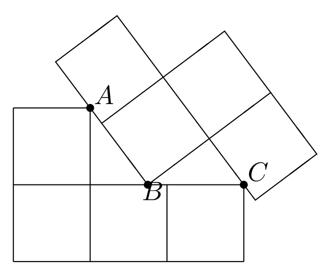
   (a) Every city lies entirely within one state.

   (b) Given any two states, the numbers of cities in them differ by at most 1.

   (c) Any city in a state shares at least one corner with the state's capital.

   What is the smallest possible number of states?

3. Eight unit squares are glued together to make two groups of four, and the groups are pressed together so as to meet in three points A, B, C as shown in the diagram. Find the distance AB.



4. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that
$$a(a-1) + b(b-1) + c(c-1) \geq 0.$$

5. The positive integers from 1 to 100 are written, in some order, in a 10×10 square. In each row, the five smallest numbers are crossed out. In each column, the five largest numbers (including those that have already been crossed out) are circled. Prove that at least 25 numbers will be circled but not crossed out.