Berkeley Math Circle
Monthly Contest 4
Due January 8, 2008

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 4
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. There are 10 bags full of coins. All coins look the same and all wight 10 grams, except the coins from one bag that are fake and all weight 9 grams. Given a scale, how could you tell which bag has the wrong coins in just one measurement? Explain your answer!

2. Find all prime numbers p such that $p^2 + 2007p - 1$ is prime as well.

   Hint. Except for 3, prime numbers are not divisible by 3. Hence if p is not equal to 3 then either $p = 3k + 1$ or $p = 3k - 1$ for some integer k. If you wish you may use lists of prime numbers from the internet (e.g. www.imomath.com/primes)

3. The sequence of numbers $1, 2, 3, \ldots, 100$ is written on the blackboard. Between each two consecutive numbers a square box is drawn. Player A starts the game and the players A and B alternate the moves. In each turn a player choses an empty box and places "$+$" or "$\cdot$" sign in it. After all the boxes are filled the expression on the blackboard is evaluated and if the result is an odd number the winner is A. Otherwise the winner is B. Determine which of the players has a winning strategy and what the strategy is.

4. The sum of the squares of five real numbers a_1, a_2, a_3, a_4, a_5 equals 1. Prove that the least of the numbers $(a_i - a_j)^2$, where $i, j = 1, 2, 3, 4, 5$ and $i \neq j$, does not exceed $1/10$.

5. Let $ABCD$ be a parallelogram. A variable line l passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centers of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line l.