# Berkeley Math Circle
# BAMO 2007 Preparation – Ivan Matić

1. Given a point $O$ and lengths $x, y, z$, prove that there exists an equilateral triangle $ABC$ for which $OA = x$, $OB = y$, $OC = z$, if and only if $x + y \geq z$, $y + z \geq x$, $z + x \geq y$ (the points $O, A, B, C$ are coplanar).

2. Given two circles of perimeter $2006$, arbitrary $2006$ points of the first circle are painted in red. On the second circle, finite number of arcs of total length less than $1$ is painted in green. Prove that it is possible to place second circle over the first one in such a way that no red point falls in green arc.

3. A polynomial $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ with integer coefficients is said to be divisible by an integer $m$ if $p(x)$ is divisible by $m$ for all integers $x$. Prove that if $p(x)$ is divisible by $m$, then $k!a_0$ is also divisible by $m$.

4. Let $ABCD$ be a regular tetrahedron and $M, N$ distinct points in the planes $ABC$ and $ADC$ respectively. Show that the segments $MN, BN, MD$ are the sides of a triangle.

5. Let $n$ be a positive integer. Prove that there exist distinct positive integers $x, y, z$ such that
$$x^{n-1} + y^n = z^{n+1}.$$

6. Prove that for every convex polygon of the area $A$ and the perimeter $P$ there exists a circle of the radius $A/P$ which is contained in the polygon.

7. Let $ABCD$ be a parallelogram. A variable line $l$ passing through the point $A$ intersects the rays $BC$ and $DC$ at points $X$ and $Y$, respectively. Let $K$ and $L$ be the centers of the excircles of triangles $ABX$ and $ADY$, touching the sides $BX$ and $DY$, respectively. Prove that the size of angle $KCL$ does not depend on the choice of the line $l$.

8. There are $n$ markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n - 1$ is not divisible by $3$.

9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following two conditions:
   (i) $f(x + f(y)) = f(x + y) + 1$ for all $x, y \in \mathbb{R}$;
   (ii) $f$ is strictly increasing (i.e. $x > y \Rightarrow f(x) > f(y)$ for all $x$ and $y$ from $\mathbb{R}$).

10. Prove that for any positive real numbers $a, b, c, d, e, f$ the following inequality holds:
$$\frac{ab}{a+b} + \frac{cd}{c+d} + \frac{ef}{e+f} \leq \frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

11. If $a$, $c$, $d$ are arbitrary positive real numbers prove that
$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}.$$

12. A circle with center $O$ passes through points $A$ and $C$ and intersects the sides $AB$ and $BC$ of the triangle $ABC$ at points $K$ and $N$, respectively. The circumscribed circles of the triangles $ABC$ and $KBN$ intersect at two distinct points $B$ and $M$. Prove that $\angle OMB = 90°$.

13. Each of cell of an $n \times n$ grid contains a real number and every two columns are different. Prove that there exists a row which can be removed so that the remaining grid still possess the property that all columns are different.

14. A finite set of unit circles is given in a plane such that the area of their union $U$ is $S$. Prove that there exists a subset of mutually disjoint circles such that the area of their union is greater that $\frac{2S}{9}$.

15. Players A and B are playing the following game on an infinite chessboard. Player A starts the game and in each of his moves he writes $X$ in some of the empty cells. In each of his moves, player B writes $O$ and the winner is the player who succeed to write his sign in $11$ consecutive cells whose centers form a line parralel to one of the lines $y = x, y = -x, y = 0, x = 0$. Prove that player B can remain in the game infinitely long without losing.