Berkeley Math Circle
Monthly Contest 8 – Solutions

1. Define

$$A = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ddots + \cfrac{1}{2006 + \cfrac{1}{2007}}}}} \quad \text{and} \quad B = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ddots + \cfrac{1}{2005 + \cfrac{1}{2006}}}}}.$$

Which of the two numbers is greater, A or B? Explain your answer!

Solution. We will determine the sign of $A - B$. If that number happens to be positive than $A > B$, otherwise $A < B$. For each n such that $1 \le n \le 2005$, let us define

$$A_n = n + \cfrac{1}{n+1 + \cfrac{1}{n+2 + \cfrac{1}{n+3 + \ddots + \cfrac{1}{2006 + \cfrac{1}{2007}}}}} \qquad B = n + \cfrac{1}{n+1 + \cfrac{1}{n+2 + \cfrac{1}{n+3 + \ddots + \cfrac{1}{2005 + \cfrac{1}{2006}}}}}$$

Then $A_1 = A$, $B_1 = B$ and we always have that $A_i - B_i$ and $A_{i+1} - B_{i+1}$ have different signs. Indeed, since $A_i = i + \frac{1}{A_{i+1}}$ and $B_i = i + \frac{1}{B_{i+1}}$ we have $A_i - B_i = \frac{B_{i+1} - A_{i+1}}{A_{i+1}B_{i+1}}$. Therefore if $A_i - B_i > 0$ then $B_{i+1} - A_{i+1} > 0$ and vice versa. Thus $A_1 - B_1$, $A_3 - B_3$, $\dots$, $A_{2005} - B_{2005}$ have the same signs. From

$$A_{2005} - B_{2005} = \left(2005 + \frac{1}{2006 + \frac{1}{2007}}\right) - \left(2005 + \frac{1}{2006}\right) < 0,$$

we get $A < B$.

2. A group of mathematicians is lost in a forest. The forest has a shape of an infinite strip that is 1 mile wide. Prove that they can choose a path that will guarantee them a way out and that is at most $2\sqrt{2}$ miles long.

Remark. The mathematicians have no device for orientation and no maps. All they know is that the forest is a region between two parallel lines 1 mile appart from each other. They can't see the end of the forest, unless they are at the edge. However, they can precisely follow any path they design, e.g. they can move along straight line, circle, etc.

Solution. Starting from their initial point A, the mathematicians should first move $\sqrt{2}$ miles in any direction. If they didn't get to the exit, they have arrived at point B. Then they should turn by 90° and walk for another $\sqrt{2}$ miles to the point C. We claim that they did reach the exit. If not, then the triangle ABC is entirely in the forest. The triangle is rectangular, isosceles and its altitude is 1 mile. Thus the triangle can't be placed in the infinite strip of width 1 unless the vertices are on the edges. Contradiction.

3. The numbers $1, 8, 4, 0$ are the first four terms of the infinite sequence. Every subsequent term of the sequence is obtained as the last digit of the sum of previous four terms. Therefore the fifth term of the sequence is 3, because $1 + 8 + 4 + 0 = 13$; the sixth term is 5 because $8 + 4 + 0 + 3 = 5$, and so on.

   (a) Will $2, 0, 0, 7$ ever appear as a subsequence?

   (b) Will $1, 8, 4, 0$ appear again as a subsequence?

   Explain your answer!

   Solution.

   (a) Yes, very soon, in fact the next four terms (from 7 to 10th) are $2, 0, 0, 7$.

   (b) We will prove that $1, 8, 4, 0$ will be a subsequence again. Assume the contrary. Since there are only finitely many combinations of four digits (precisely 10^4), and the sequence is infinite, some combination of four digits (say (a, b, c, d)) has to reappear. Assume that $(x_n, x_{n+1}, x_{n+2}, x_{n+3}) = (a, b, c, d)$ is the first occurence of (a, b, c, d) and that $(x_m, x_{m+1}, x_{m+2}, x_{m+3}) = (a, b, c, d)$ is the second. Clearly $m > n$. However, x_{n-1} and x_{m-1} are uniquely determined and they have to be the same numbers. Thus $x_{n-1} = x_{m-1}$, $x_{n-2} = x_{m-2}$, and so on. This means that $x_1 = x_{m-n+1}$, $x_2 = x_{m-n+2}$, $x_3 = n_{m-n+3}$, and $x_4 = x_{m-n+4}$, which is a contradiction.

4. Let X, Y, and Z be the points on the sides BC, CA, and AB of the triangle ABC, such that $\triangle XYZ \sim \triangle ABC$ ($\sphericalangle X = \sphericalangle A$, $\sphericalangle Y = \sphericalangle B$). Prove that the orthocenter of $\triangle XYZ$ coincides with the circumcenter of $\triangle ABC$.

   Solution. Let x, y, and z be the points passing through X, Y, and Z, parallel to YZ, ZX, and XY. Let P, Q, and R be the intersection point of the lines y and z; z and x; x and y, respectively. Then we have $\triangle PQR \sim \triangle XYZ$. The points X, Y, and Z are respectively the midpoints of RQ, QP, and PR. Let M be the orthocenter of $\triangle XYZ$. Obviously, M is the circumcenter of $\triangle PQR$ and $\sphericalangle ZMY = 180° - \sphericalangle X = 180° - \sphericalangle P = 180° - \sphericalangle A$. Hence the points P, A, Z, M, and Y belong to a circle. In a similar way we prove that the points Z, B, R, X and M belong to a circle. Then $\sphericalangle PMA = \sphericalangle PZA = \sphericalangle BZR = \sphericalangle BMR$. Since $MR = MP$ and $\sphericalangle PAM = \sphericalangle BRM = 90°$, we conclude that $MA = MB$. Analogously we conclude that $MA = MC$ implying that M is the circumcenter of $\triangle ABC$.

5. Let $n > 1$ be an odd integer. Prove that every integer l satisfying $1 \leq l \leq n$ can be represented as a sum or difference of two integers each of which is less than n and relatively prime to n.

   Solution. We will use the following lemma (it is known as The Chinese Remainder Theorem).

   Lemma. Let $m_1, m_2, \ldots, m_k$ be different relatively prime numbers. If $q_1, q_2, \ldots, q_k$ are arbitrary non-negative integers then there exists a natural number x less than $m_1 m_2 \ldots m_k$ such that

$$\begin{aligned} x &\equiv q_1 \pmod{m_1} \\ x &\equiv q_2 \pmod{m_2} \\ &\vdots \\ x &\equiv q_k \pmod{m_k}. \end{aligned}$$

   Proof of the Lemma. We can assume that $q_i < m_i$ for $1 \leq i \leq k$. We will prove this by induction. For $k = 2$ we consider the numbers $x_1 = q_1$, $x_2 = m_1 + q_1$, $x_3 = 2m_1 + q_1$, $\ldots$, $x_{m_2} = (m_2 - 1)m_1 + q_1$. If two of the number x_i and x_j give the same remainder upon division by m_2 we have that $x_j - x_i$ is divisible by m_2 which is impossible since that difference is equal to $(j - i)m_1$. Thus all x_is give different remainders modulo m_2 and one of them has to give a remainder q_2. This finishes the proof for the case $k = 2$.

   Assume now that the statement holds for k and we want to prove it for $k + 1$. Applying the inductional hypothesis $q_1, \ldots, q_k$ we find a number x' such that $x' \equiv q_i \pmod{m_i}$ for $1 \leq i \leq m_i$. Now by the case $k = 2$ we get that there is an x such that $x \equiv x' \pmod{m_1 \cdots m_k}$ and $x \equiv q_{k+1} \pmod{m_{k+1}}$. Such x satisfies the required condition. $\square$

   Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are prime numbers and $\alpha_1, \ldots \alpha_k$ positive integers. Assume that l is given and that $l \equiv t_i \pmod{p_i}$. Since $p_i > 2$ we can choose s_i such that $s_i \not\equiv 0 \pmod{p_i}$ and $s_i \not\equiv t_i \pmod{p_i}$. By the Chinese Remainder Theorem there exists an integer $s < p_1 \ldots p_k$ such that $s \equiv s_i \pmod{p_i}$ for every $i = 1, \ldots, k$. If $s < l$ then choose $a = s$ and $b = l - s$. If $s > l$ then we can choose $a = s$ and $b = s - s$. It is easy to verify that such a and b satisfy the conditions of the problem.