Berkeley Math Circle
Monthly Contest 7
Due April 3, 2007

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 7
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find all positive integers n such that $n(n+1)$ is a perfect square.

2. Prove that
$$A = \sqrt{4-2\sqrt{3}} - \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
is an integer.

3. An isolated island has the shape of a circle. Initially there are 9 flowers on the circumference of the island: 5 of the flowers are red and the other 4 are yellow. During the summer 9 new flowers grow on the circumference of the island according to the following rule: between 2 old flowers of the same color a new red flower will grow, between 2 old flowers of different colors, a new yellow flower will grow. During the winter, the old flowers die, and the new survive. The same phenomenon repeats every year.

   Is it possible (for some configuration of initial 9 flowers) to get all red flowers after finitely many years?

4. Given a triangle ABC, a circle k is tangent to the lines AB and AC at B and P. Let H be the foot of perpendicular from the center O of k to BC, and let T be the intersection point of OH and BP. Prove that AT bisects the segment BC.

5. A society has 100 members and every two members are either friends or enemies. Prove that there are two members of the society that have an even number of common enemies.