# Berkeley Math Circle
# Monthly Contest 6 – Solutions

1. If $a$ and $b$ are two positive numbers not greater than $1$ prove that

$$\frac{a+b}{1+ab} \le \frac{1}{1+a} + \frac{1}{1+b}.$$

When does the equality hold?

**Solution.** If we multiply both sides of the inequality with $(1+ab)(1+a)(1+b)$ the required inequality becomes equivalent to

$$(a+b)(1+a)(1+b) \le (1+a)(1+ab) + (1+b)(1+ab) \Leftrightarrow$$

$$(a+b)(1+a+b+ab) \le 1+ab+a+a^2b+1+ab+b+ab^2 \Leftrightarrow$$

$$a+a^2+ab+a^2b+b+ab+b^2+ab^2 \le 1+ab+a+a^2b+1+ab+b+ab^2 \Leftrightarrow$$

$$a^2+b^2 \le 2.$$

The last inequality is true and the equality holds if and only if $a=b=1$.

2. A car is moving at a constant speed. Every $15$ minutes it makes a $90^\circ$ turn to either left or right. If the car has started the trip at some point $A$, prove that it can return to $A$ only after an integer number of hours.

**Solution.** It is important to notice that an hour consists of $60$ minutes and $60/15 = 4$. That's why it is common to say that $15$ minutes is a "quarter of an hour". We may assume that the car is traveling in a square grid where the size of each square is the distance that the car travels in $15$ minutes. Assume further that we know what is "north", "east", "west", and "south". Let $n$, $e$, $w$, and $s$, denote respectively the number of north, east, west, and south moves. If the car has returned to $A$, then $n = s$ and $w = e$. However since the car at every vertex changes the direction from the set $\{$north, south$\}$ to the set $\{$east, west$\}$, or vice versa, we see that $n = w$. Thus $n = s = w = e$ and the total number of moves is $n+s+w+e = 4n$ which is divisible by $4$. Thus the total time of travel is $4 \cdot 15'n = n$ hours.

3. Let $M$ be an interior point of a parallelogram $ABCD$. Prove that $MA+MB+MC+MD$ is strictly less than the length of the perimeter of $ABCD$.

**Solution.** Denote by $X$ and $Y$ the points of intersection of the segments $AB$ and $CD$ with the line through $M$ parallel to $BC$. Similarly, let $U$ and $V$ denote the points of intersection of the segments $AD$ and $BC$ with the line through $M$ parallel to $AB$. Then $MA < AU + UM = XM + UM$, $MB < MX + XB = MX + MV$, $MC < MV + VC = MV + MY$, and $MD < MY + YD = MY + MU$. Hence $MA+MB+MC+MD < 2(XM+YM)+2(UM+VM) = 2AD+2AB = AB+BC+CD+DA$.

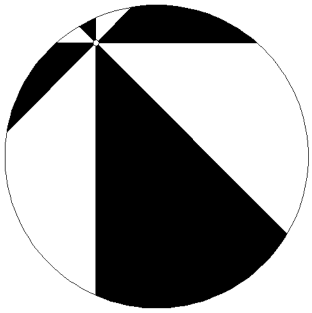
4. Prove that the product of $6$ consecutive positive integer is never equal to $n^5$ for some positive integer $n$.

**Solution.** It is easy to show that at least one of $6$ consecutive numbers is relatively prime to each of the others ($3$ of these numbers are odd, exactly one of these odd numbers is divisible by $3$, and for each prime $p > 3$ at most one is divisible by $p$). Assume that $x-2, x-1, x, x+1, x+2, x+3$, $x \ge 3$ are given numbers. The number relatively prime to the others has to be a perfect $5$th power. Let $F$ be the product of the other $5$ numbers. We have:

$$P(x) = (x-2)(x-1)x(x+1)(x+2) = x^5-5x^3+4x \le F \le x^5+5x^4+5x^3-5x^2-6x = (x-1)x(x+1)(x+2)(x+3) = Q(x).$$

Since for $x \ge 3$ we have $(x+1)^5 > Q(x) \ge F \ge P(x) > (x-1)^5$ we conclude that $F = x^5$. However this is a contradiction since $x$ is relatively prime to $x-1$ and $x+1$ and at least one of these numbers is a factor of $F$.

5. The point $K$ lies in the interior of the unit circle. Four lines are drawn through $K$ such that each two adjacent lines form an angle of $45°$, as shown in the picture. In such a way the circle is divided into $8$ regions. Four of these regions are colored such that no two colored regions share more than one point. Find the maximal and minimal possible value for the total area of colored regions.



**Solution.**

We will show that the total area of colored regions is equal to $\frac{1}{2}$ the area of the circle no matter how the lines are drawn.

First we will inscribe our circle in a square whose sides are parallel to 2 of these lines. If we color the regions of the square in the same way as the regions of the circle are colored we can easily prove that the total area of colored regions is equal to the area of non-colored regions for the square. Hence it remains to prove the equality of the areas of colored and uncolored regions that are outside the cirle. Now we will look at the pictures below. For each picture it is obvious that it has the same area of colored parts as the previous picture (as some colored parts are just moved from one location to the other).