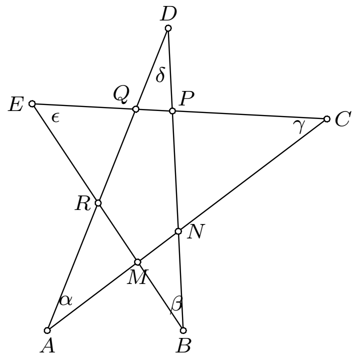
Berkeley Math Circle
Monthly Contest 5 – Solutions

1. Do there exist 100 consecutive positive integers such that their sum is a prime number?

   Hint. What is the method for summing 100 consecutive numbers?

   Solution. No. Let $n, n+1, \dots, n+99$ be any 100 consecutive positive integers. Then $n+(n+1)+(n+2)+\cdots+(n+99) = 100n+(1+2+\cdots+99)$. However, $1+2+\cdots+99 = (1+99)+(2+98)+(3+97)+\cdots+(49+51)+50 = 49\cdot 100+50 = 50(2\cdot 49+1) = 50\cdot 99$. Thus $n+(n+1)+\cdots+(n+99) = 100n+50\cdot 99 = 50(2n+99)$ and this is not prime.

2. Let $ABCDE$ be a convex pentagon. If $\alpha = \angle DAC$, $\beta = \angle EBD$, $\gamma = ACE$, $\delta = \angle BDA$, and $\epsilon = \angle BEC$, as shown in the picture, calculate the sum $\alpha+\beta+\gamma+\delta+\epsilon$.

   Solution. Let M, N, P, Q, R denote the intersections of the lines AC and BE, AC and BD, CE and BD, DA and CE, EB and AE, respectively. Since the sum of the internal angles of a triangle is $180°$, from $\triangle AMR$ we get $\alpha = 180° - \angle AMR - \angle MRA$. We also know that $\angle AMR = \angle BMN$ and we can denote these angles by $\angle M$. Analogously $\angle ARM = \angle ERQ = \angle R$. With similar equations for β, γ, δ, and ϵ we get that $\alpha+\beta+\gamma+\delta+\epsilon = 5\cdot 180° - 2(\angle M+\angle N+\angle P+\angle Q+\angle R)$. Since $\angle M$, $\angle N$, $\angle P$, $\angle Q$, and $\angle R$ are exterior angles of the pentagon $MNPQR$ their sum has to be $360°$ implying that $\alpha+\beta+\gamma+\delta+\epsilon = 180°$.

3. Bart has 17 and 19 dollar bills only.

   (a) Prove that these bills are fake.

   (b) Prove that there exists $m > 0$ such that for each $n \geq m$ Bart can give to Lisa exactly n dollars using his bills.

   Remark. Part (a) is worth 0 points but we would like to see your "proof".

   Solution. We can have $m = 17\cdot 19$. For each $n > 17\cdot 19$, we consider the set $S = \{n, n-17, n-2\cdot 17, \dots, n-18\cdot 17\}$. If none of these numbers is divisible by 19, two of them will give the same residue upon division by 19. Indeed, there are 19 numbers and without 0 there are only 18 residues. Let $n-17a$ and $n-17b$ give the same residue mod 19. Assuming $a < b$ we get that $n-17a-(n-17b) = 17(b-a)$ is divisible by 19 which is impossible.

   Hence one of the numbers from S is divisible by 19, say $n-k\cdot 17$. Then $n-k\cdot 17 = l\cdot 19$ and $n = k\cdot 17+l\cdot 19$ which solves the problem.

4. Does there exist a convex polygon that can be partitioned into non-convex quadrilaterals?

   Solution. The answer is no. Assume that, on the contrary it is possible to partition a polygon P into non-convex quadrilaterals. Let n be the number of quadrilaterals. Denote by S the total sum of all internal angles of all the quadrilaterals. Since the sum of internal angles of each quadrilateral is $360°$ we have $S = 360°$. However, each of the nonconvex angles has to be in the interior of P, hence the sum of angles around the vertex of that angle has to be $360°$. This immediately gives $360°n$ as the sum of angles around such vertices. Since those are not the only vertices (at least the vertices of P will contribute to the sum S), we have that $S > 360°$ and this is a contradiction.

5. The numbers from the table

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

   satisfy the inequality

$$\sum_{i=1}^{n} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \leq M,$$

   for every choice $x_i = \pm 1$. Prove that $|a_{11}| + |a_{22}| + \cdots + |a_{nn}| \leq M$.

Solution. We will sum the given inequality over all possible choices for $(x_1, \dots, x_n)$. Since the number of such choices is 2^n we obtain

$$\sum_{i=1}^{n} \sum_{(x_1,\dots,x_n)} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \leq 2^n M. \tag{1}$$

Now we will consider the expressions $\sum_{(x_1,\dots,x_n)} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n|$. We will group summands in pairs – members of each pair will have all terms x_j the same, except for the i-th. One member will have $x_i = 1$ and the other $x_i = -1$. To each of those terms we will apply the inequality $|a_{ii} + B| + |a_{ii} - B| \geq |2a_{ii}|$. More precisely,

$$\begin{aligned}
& \sum_{(x_1,\dots,x_n)} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \\
= & \sum_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)} \left(|a_{i1}x_1 + a_{i2}x_2 + \cdots + a_ii + \cdots + a_{in}x_n| + |a_{i1}x_1 + a_{i2}x_2 + \cdots - a_ii + \cdots + a_{in}x_n|\right) \\
\geq & \sum_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)} |2a_{ii}| = 2^n |a_{ii}|.
\end{aligned}$$

Now (1) implies that $2^n M \geq 2^n(|a_{11}| + |a_{22}| + \cdots + |a_{nn}|)$ which is equivalent to the inequality we have to prove.