Berkeley Math Circle
Monthly Contest 4 – Solutions

1. Do there exist positive integers x, y, and z such that $x^{2006} + y^{2006} = z^{2007}$? Explain your answer.

   Remark. If the answer is *yes*, you should give an example of such x, y, and z. If the answer is *no*, you should prove that no x, y, z can satisfy the above equation.

   Solution. Yes, for example, $x = y = z = 2$.

2. Let O be the intersection of the diagonals AC and BD of the convex quadrilateral $ABCD$. Let S_1, S_2, S_3, and S_4 denote the areas of the triangles ABO, BCO, CDO, and DAO.

   (a) Prove that $S_1 \cdot S_3 = S_2 \cdot S_4$.

   (b) Does there exist a quadrilateral $ABCD$ such that S_1, S_2, S_3, and S_4 are consecutive positive integers in some order?

   Solution.

   (a) Let M and N be feet of perpendiculars from B and D to AC. Then $S_1 = AO \cdot BM/2$, $S_2 = CO \cdot BM/2$, $S_3 = CO \cdot DN/2$, and $S_4 = AO \cdot DN/2$. Now the desired statement follows immediately from the previous four relations.

   (b) We will prove that the answer to the question is no. Assume the opposite, i.e. that n, $n+1$, $n+2$, $n+3$ are the given areas, where n is a positive integer. Since $S_1 \cdot S_3 = S_2 \cdot S_4$ and $n < n+1 < n+2 < n+3$ we must have $n(n+3) = (n+1)(n+2)$ which is impossible.

3. 2006 vertices of a regular 2007-gon are red. The remaining vertex is green. Let G be the total number of polygons whose one vertex is green and the others are red. Denote by R the number of polygons whose all vertices are red. Which number is bigger, R or G? Explain your answer.

   Solution. We will prove that $G \geq R$. For each polygon $\mathcal{P}$ with all red vertices we can correspond a polygon with one green vertex (namely we can add the green vertex to the set of vertices of $\mathcal{P}$). Thus $G \geq R$. However, $G > R$ since the triangles with one green vertex can't be corresponded to some polygon whose all vertices are red.

4. A sequence of numbers $\{a_n\}$ is given by $a_1 = 1$, $a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}$ for $n \geq 1$. Prove that each term of the sequence is an integer.

   Solution. From the definition of the sequence we see that $(a_{n+1} - 2a_n)^2 = 3a_n^2 + 1$. After simplification we get

   $$a_{n+1}^2 + a_n^2 - 4a_n a_{n+1} = 1.$$

   Adding $3a_{n+1}^2$ to both sides of the last equation gives us $(2a_{n+1} - a_n)^2 = 3a_{n+1}^2 + 1$ and after taking the square root of both sides we get

   $$\sqrt{3a_{n+1}^2 + 1} = 2a_{n+1} - a_n. \tag{1}$$

   On the other hand, from the definition of the sequence we see that

   $$\sqrt{3a_{n+1}^2 + 1} = a_{n+2} - 2a_{n+1}. \tag{2}$$

   From (1) and (2) we conclude that $a_{n+2} = 4a_{n+1} - a_n$ wich together with $a_1 = 1$, $a_2 = 4$ implies that all terms are integers.

5. A finite set of circles in the plane is called *nice* if it satisfies the following three conditions:

   (i) No two circles intersect in more than one point;

   (ii) For every point A of the plane there are at most two circles passing through A;

   (iii) Each circle from the set is tangent to exactly 5 other circles form the set.

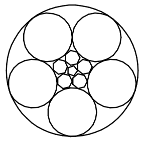
   Does there exist a nice set consisting of exactly

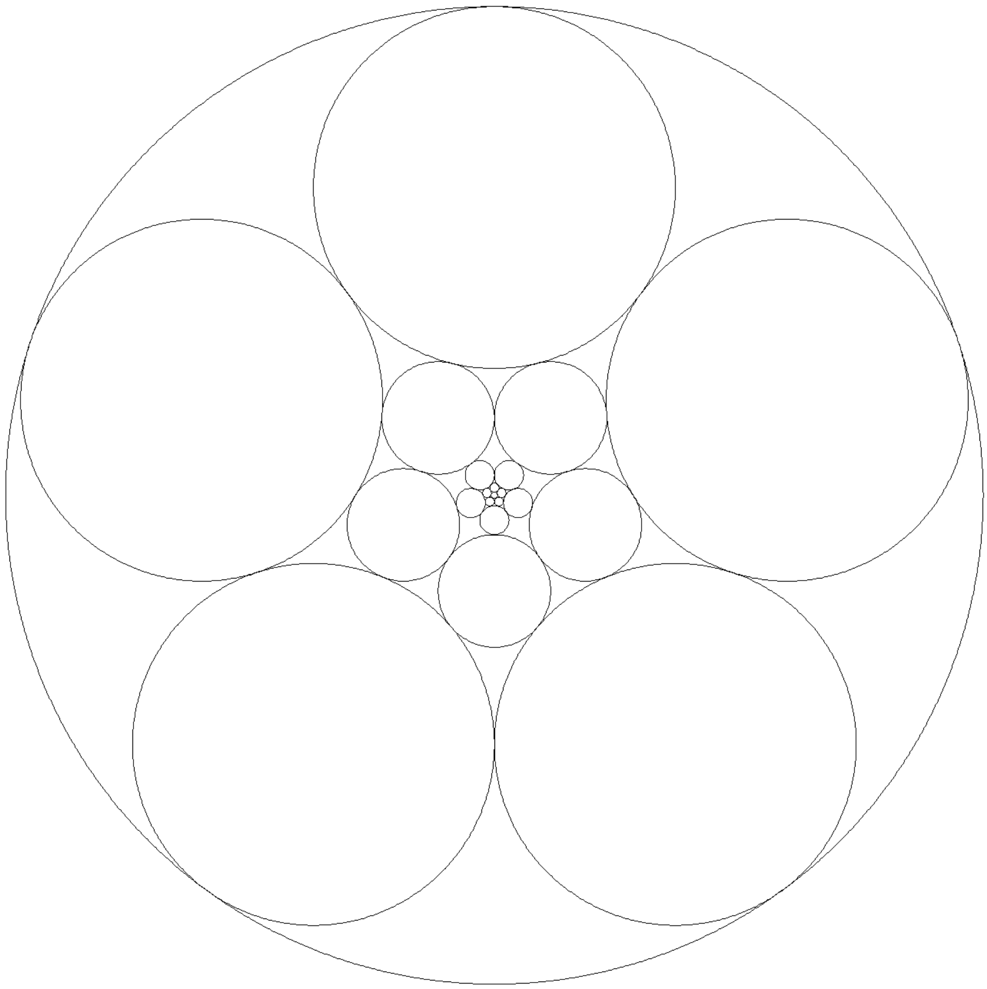
   (a) 2006 circles?

   (b) 2007 circles?

Solution.

(a) The answer is yes. The following picture shows that there is a nice set consisting of exactly 12 circles. It is also possible to construct a nice set with 22 circle (the picture can be found below). Since $2006 = 158 \cdot 12 + 5 \cdot 22$ we can make a nice set of 2006 by making a union of 158 discjoint nice sets of 12 circles each, and 5 discjoint nice sets each of which contains 22 circles.





(b) We will prove that nice set can't contain odd number of circles. Let n be a total number of circles. We will count the number of pairs (k, P) where k is a circle in the set and P a point at which k touches another circle. For each circle k there are exactly 5 such pairs, and hence the total number of pairs is $5n$. For each point P there are exactly 2 pairs corresponding to it. Hence the total number of pairs has to be even, but $5n$ can't be even if n is odd.