Berkeley Math Circle
Monthly Contest 2 – Solutions

1. Find all positive prime numbers p and q such that $p^2 - q^3 = 1$.

   Remark. p is prime if it has only two divisors: 1 and itself. The numbers $2, 3, 5, 7, 11, 13$ are prime, but $1, 4, 6, 8, 9$ are not.

   Solution. If both of the numbers p and q are odd, then each of p^2 and q^3 has to be odd so their difference must be even. Hence, at least one of p and q has to be even. Since 2 is the only even prime number, and at the same time it is the smallest prime number, we must have $q = 2$. Now we get $p = 3$.

2. A line l and two points A and B are given in a plane in such a way that A belongs to l but B doesn't. Construct the circle k that passes through B and touches l at the point A.

   Solution. Let m be the bisector of the segment AB. The center of the circle k has to belong to m. Similarly, since the circle has to be tangent to l at the point A its center has to be located on the line a perpendicular to l that passes through A.

   The lines m and a are easy to construct and their intersection is the center O of the circle. Now we have the center and the point B hence the circle is determined.

3. If the sum of digits in a decimal representation of a natural number n is equal to 2006, prove that n can't be a perfect square of an integer.

   Solution. The remainder of n upon division by 3 is equal to the sum of its digits, i.e. 2006. Hence number n has a remainder 2 upon division by 3 and no square can have that remainder.

4. Let $\triangle ABC$ be a triangle such that $\angle A = 90^\circ$. Determine whether it is possible to partition $\triangle ABC$ into 2006 smaller triangles in such a way that

   1° Each triangle in the partition is similar to $\triangle ABC$;

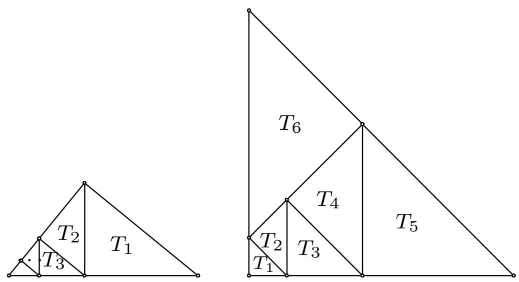
   2° No two triangles in the partition have the same area.

   Explain your answer!

   Solution. The required partition is always possible. We consider 2 cases:

   (a) *The triangle is not isosceles* – First we construct the perpendicular from the vertex of the right angle. The triangle is divided into two similar, but non-congruent triangle. Now we divide smaller triangle, and keep going until the total number of triangle becomes 2006.

   (b) *The triangle is isosceles* – Repeat the procedure from the previous problem until we get 2001 triangles. Then we divide one of the smallest triangles into 6 smaller and noncongruent triangles as shown in the second picture.



5. Let $S > 0$. If a, b, c, x, y, z are positive real numbers such that $a + x = b + y = c + z = S$, prove that

$$ay + bz + cx < S^2.$$

   Solution. Denote $T = S/2$. One of the triples (a, b, c) and (x, y, z) has the property that at least two of its members are greater than or equal to T. Assume that (a, b, c) is the one, and choose $\alpha = a - T$, $\beta = b - T$, and $\gamma = c - T$. We then have $x = T - \alpha$, $y = T - \beta$, and $z = T - \gamma$. Now the required inequality is equivalent to

$$(T + \alpha)(T - \beta) + (T + \beta)(T - \gamma) + (T + \gamma)(T - \alpha) < 4T^2.$$

   After simplifying we get that what we need to prove is

$$-(\alpha\beta + \beta\gamma + \gamma\alpha) < T^2. \qquad (1)$$

   We also know that at most one of the numbers α, β, γ is negative. If all are positive, there is nothing to prove. Assume that $\gamma < 0$. Now (1) can be rewritten as $-\alpha\beta - \gamma(\alpha + \beta) < T^2$. Since $-\gamma < T$ we have that $-\alpha\beta - \gamma(\alpha + \beta) < -\alpha\beta + T(\alpha + \beta)$ and the last term is less than T since $(T - \alpha)(T - \beta) > 0$.