Berkeley Math Circle

Monthly Contest 6
Due March 14, 2006

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 6
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Find all pairs (m, n) of natural numbers such that $200m + 6n = 2006$.

2. Circles k_1 and k_2 intersect at points A and B. Line l is the common tangent to these circles and it touches k_1 at C and k_2 at D such that B belongs to the interior of the tirangle ACD. Prove that $\angle CAD + \angle CBD = 180°$.

3. If x and y are two positive numbers less than 1, prove that
$$\frac{1}{1-x^2} + \frac{1}{1-y^2} \geq \frac{2}{1-xy}.$$

4. If n is natural number such that $2n+1$ and $3n+1$ are perfect squares, prove that $5n+1$ can't be a prime number.

5. Every two members of a certain society are either *friends* or *enemies*. Suppose that there are n members of the society, that there are exactly q pairs of friends, and that in every set of three persons there are two who are enemies to each other. Prove that there is at least one member of the society among whose enemies there are at most $q \cdot \left(1 - \frac{4q}{n^2}\right)$ pairs of friends.