Berkeley Math Circle
Monthly Contest 2
Due November 29, 2005

Instructions

This contest consists of 5 problems, some of which are easier than the others. Every problem is worth 7 points. Please, write solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade and school, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
in grade 5
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A frog is jumping on a 8×8 chessboard. At each step the frog jumps from one unit square to one of the squares that is adjacent to the previous position of the frog (squares are *adjacent* if they share an edge). Is it possible for frog to start from the lower-left corner of the chessboard, visit each unit square exactly once, and finish its trip at the upper-right corner of the chessboard? Justify your answer!

2. Let $\triangle ABC$ be a triangle such that $\angle ACB = 90^\circ$ and $AC < BC$. Let k be the circle with diameter AC and let E be the intersection of k with AB. Denote by t a tangent line to k that contains point E and let D be the intersection of t with BC. Prove that the triangle $\triangle BDE$ is isosceles.

   Hint: Is there some other right triangle in the picture? Recall the properties of the midpoint of hypothenuse of a right triangle.

3. If a, b, c are positive real numbers that satisfy $a^2 + b^2 + c^2 = 1$, find the minimal value of

$$S = \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}.$$

   Hint: The famous inequality $x^2 + y^2 \geq 2xy$ is helpful here.

4. The set $\{1, 2, \ldots, 2004\}$ contains $1002 + k$ numbers that are colored in blue ($1 \leq k \leq 1002$). Prove that we can find $2k$ blue numbers whose sum is divisible by 2005.

   Hint: If $n > m$ pigeons are put into m pigeonholes, there's a hole with more than one pigeon...

5. Every point in the plane is painted in blue or red. Prove that there are either two blue points at the distance exactly one, or 4 colinear red points A_1, A_2, A_3, A_4 such that $A_1A_2 = A_2A_3 = A_3A_4 = 1$.