Bay Area Mathematical Olympiad
and Mathematical Circles
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Berkeley Math Circle
Monthly Contest 2
Solutions

1. Solve

$$2\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)(x+5)}}}} = x$$

As the left hand side is nonnegative, we see that any solution will have $x \geq 0$. For such x we have $\sqrt{1+(x+3)(x+5)} = \sqrt{x^2+8x+16} = \sqrt{(x+4)^2} = |x+4| = x+4$. Proceeding similarly we get

$$\begin{aligned} & 2\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+(x+3)(x+5)}}}} \\ = \quad & 2\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}} \\ = \quad & 2\sqrt{1+x\sqrt{1+(x+1)(x+3)}} = 2\sqrt{1+x(x+2)} = 2(x+1) \end{aligned}$$

Solving $2(x+1) = x$ gives $x = -2$, which is negative. Therefore the equation has no solutions.

2. The circle ω passes through the vertices A and B of a unit square $ABCD$. It intersects AD and AC at K and M respectively. Find the length of the projection of KM onto AC.

Let T be the point of intersection of ω with BC. Then, as $\angle ABT$ a right angle, AT is a diameter, and $\angle AMT$ is also a right angle. Therefore the projections of KM and KT on AC coincide. But the length of the projection of KT is $\frac{\sqrt{2}}{2}$ because the length of KT is one, and the angle between KT and AC is 45.

3. A king is placed in the left bottom corner of the 6 by 6 chessboard. At each step it can either move one square up, or one square to the right, or diagonally - one up and one to the right. How many ways are there for the king to reach the top right corner of the board?

We shall make a 6×6 table. In each cell of the table we will write a number of ways in which the king can reach that cell. We will fill it out gradually - starting with a row of ones at the bottom and a column of ones at the left. To fill out the rest we use the following rule: the number in each cell is equal to the sum of the numbers immediately below, to the left, and diagonally (to the left and below). The result is:

1	11	61	231	681	1683
1	9	41	129	321	681
1	7	25	63	129	231
1	5	13	25	41	61
1	3	5	7	9	11
1	1	1	1	1	1

The answer is 1683.

4. In the triangle ABC the angle B is not a right angle, and $AB : BC = k$. Let M be the midpoint of AC. The lines symmetric to BM with respect to AB and BC intersect AC at D and E. Find $BD : BE$.

As BC is the angle bisector in the triangle MBE, we have $\frac{CE}{BE} = \frac{CM}{BM}$ (by a well-known property of the angle bisector). Similarly, $\frac{AD}{BD} = \frac{AM}{BM}$. Draw a line BM' symmetric to BM with respect to the angle bisector of ABC (point M' is on the line AC). BM' bisects the angle DBE. Using the same property of the angle bisector, we get $\frac{EM'}{BE} = \frac{DM'}{BD}$. Subtracting from this $\frac{CE}{BE} = \frac{CM}{BM}$ we get $\frac{CM'}{BE} = \frac{AM'}{BD}$ or $\frac{BD}{BE} = \frac{AM'}{CM'}$.

Now it remains only to find the ratio in which M' divides AC. To do that, note that MBC and MBA have equal areas: $\frac{1}{2}BM \cdot BC \cdot \sin\angle MBC = \frac{1}{2}BM \cdot BA \cdot \sin\angle MBA$. Therefore $\frac{\sin\angle MBC}{\sin\angle MBA} = \frac{AB}{BC} = k$. Hence

$$\begin{aligned}\frac{AM'}{CM'} = \frac{S_{ABM'}}{S_{BCM'}} &= \frac{\frac{1}{2}BM' \cdot BA \cdot \sin\angle M'BA}{\frac{1}{2}BM' \cdot BC \cdot \sin\angle M'BC} \\ &= k \cdot \frac{\sin\angle M'BA}{\sin\angle M'BC} = \frac{\sin\angle MBC}{\sin\angle MBA} = k \cdot k = k^2\end{aligned}$$

5. One marks 16 points on a circle. What is the maximum number of acute triangles with vertices in these points?

Consider the set of all angles $M_1M_2M_3$, where M_1,M_2 and M_3 is an arbitrary triple of selected points. There are $\frac{16\cdot15\cdot14}{2} = 1680$ different angles in this set. Suppose n of them are not acute. We shall prove $n \geq 392$. For each integer m between 1 and 7, take a chord with endpoints among the selected points such that there are exactly m selected points to one side of the chord (not including the endpoints). We will call such a chord an *m-chord*. Each m-chord subtends not less than m nonacute angles with vertices among the marked points. For each $m \leq 6$ there are exactly 16 m-chords, and for $m = 7$ there are exactly 8 of them. So the total number of nonacute angles is at least $16(1 + 2 + \ldots + 6) + 8 * 7 = 392$.

There are $\frac{16 \cdot 15 \cdot 14}{6} = 560$ triangles with vertices among the marked points. Each nonacute angle will "spoil" exactly one triangle, so the number of acute triangles is not greater than $560 - 392 = 168$.

It is only left to construct an example with exactly 168 acute triangles. Mark eight consecutive vertices of a right 16-gon $V_1, \ldots V_8$. Draw a line through the center of the 16-gon not parallel to V_1V_8, such that all V's lie on the same side of that line (and not on the line). Reflecting the $V's$ with respect to that line we get points $V_1{}', \ldots V_8{}'$. We claim that the set $V_1, \ldots V_8, V_1{}' \ldots V_8{}'$ is as wanted. Indeed, there are no diametrically opposite points in this set (otherwise we would get $V_1V_1{}' = 0$ or $V_8V_8{}' = 0$), and so for $m = 7$ each m-chord subtends exactly 8 nonacute angles. Moreover, for each $m \leq 6$ each m-chord subtends exactly m nonacute angles, and so $n = 392$, and the number of acute triangles is 168.