Bay Area Mathematical Olympiad
and Mathematical Circles

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Berkeley Math Circle
Monthly Contest 1
Due October 19, 2003

1. a) One Sunday, Zvezda wrote 14 numbers in a circle, so that each number is equal to the sum of its two neighbors. Prove that the sum of all 14 numbers is 0.

b) On the next Sunday, Zvezda wrote 21 numbers in a circle, and this time each number was equal to **half** the sum of its two neighbors. What is the sum of all 21 numbers, if one of the numbers is 3?

a) Denoting the numbers $a_1, a_2, \ldots, a_{14}$, and their sum as S we have $a_i = a_{i-1} + a_{i+1}$ for $i = 1, \ldots, 14$ (we take $a_{15} = a_1$, $a_0 = a_{14}$). Summing all these equalities we get $S = 2S$ (since each a_i appears exactly once on the left and exactly twice on the right). Therefor $S = 0$.

b) Again, let the numbers be $a_1, \ldots, a_{21}$. Let a_i be maximal among the numbers. Then we have $a_i \geq a_{i-1}$, $a_i \geq a_{i+1}$ and so $a_i \geq \frac{a_{i-1}+a_{i+1}}{2}$. But $a_i = \frac{a_{i-1}+a_{i+1}}{2}$, so both inequalities are actually equalities. Next, considering now a_{i-1} we conclude $a_{i+1} = a_{i+2}$. Proceeding in this way, we conclude that all the numbers are equal. As one of them is 3, the sum of all numbers is $3 * 21 = 63$.

2. A grasshopper lives on a coordinate line. It starts off at 1. It can jump either 1 unit or 5 units either to the right or to the left. However, the coordinate line has holes at all points with coordinates divisible by 4 (e.g. there are holes at -4, 0, 4, 8 etc.), so the grasshopper can not jump to any of those points. Can it reach point 3 after 2003 jumps?

Each jump changes the parity of grasshopper's coordinate. After 2003 jumps the grasshopper will be at an even point on the coordinate line, and therefore can not be at 3.

3. The sets A and B and be form a *partition* of positive integers if $A \cap B = \emptyset$ and $A \cup B = N$. The set S is called *prohibited* for the partition, if $k + l \neq s$ for any $k, l \in A$, $s \in S$ and any $k, l \in B$, $s \in S$.

a) Define *Fibonacci numbers* f_i by letting $f_1 = 1, f_2 = 2$ and $f_{i+1} = f_i + f_{i-1}$, so that $f_3 = 3$, $f_4 = 5$ etc. How many partitions for with the set F of all Fibonacci numbers is prohibited are there? (We count A, B and B, A as the same partition.)

b) How many partitions for which the set P of all powers of 2 is prohibited are there? What if we require in addition that $P \subseteq A$?

b) We prove the following: Given a partition of the set of all powers of 2 (i.e. two sets Q and R such that each 2^k is in exactly one of Q, R) there exists unique partition A, B of positive integers with all powers of 2 prohibited and with $Q \subseteq A$, $R \subseteq B$. Note that this implies that there are infinitely many partitions of integers with powers of 2 prohibited, and exactly one such partition with $P \subseteq A$.

First we show that required partition is unique if it exists. Without loss of generality 1 is in A. Suppose now that we have been able to place all integers up to some k unambiguously, i.e. for any $l < k$ we know whether l is in A or in B. If k is a power of 2 we know where to put it. Otherwise let 2^i be the smallest power of 2 strictly greater than k. Then $d = 2^i - k$ is positive and less than k Therefore we know to which of two sets (A or B)the number d belongs. Then we have no choice but to place k in the other set (otherwise k and d would be in the same set, and since $k + d = 2^i$ this would contradict A, B being acceptable). So there exists no more than one acceptable partition.

We now show that desired partition exists. Since we have already defined above a recursive construction producing A, B, we just need to check that the resulting partition is in fact acceptable. Suppose not. Then there exist $n < m$ such that $n + m = 2^i$ for some i and m, n are in the same set of the partition. Then m is not a power of 2 (if it were, n would be greater than m). Note that $n < m$ implies that 2^i is the smallest power of 2 strictly greater than m. Then by construction m is in the set different from n. Contradiction. So there is no such pair m, n. The partition constructed above works. This completes the proof.

a) The proof is similar to that of part b. We will call a partition acceptable if F is prohibited for it. First, we show that there exists no more than one acceptable partition. Since we do not distinguish between A, B we may assume without loss of generality that 1 is in A. Suppose now that we have been able to place all integers up to some k unambiguously, i.e. for any $l < k$ we know whether l is in A or in B. Let f_i be the smallest Fibonacci number strictly greater than k. Then $d = f_i - k$ is positive and less than k (to see that, note that if $f_i - k > k$ then $2k < f_i$, $k < f_i/2 \leq f_{i-1}$, contradicting the choice of f_i).

Therefore we know to which of two sets (A or B)the number d belongs. Then we have no choice but to place k in the other set (otherwise k and d would be in the same set, and since $k+d=f_i$ this would contradict A, B being acceptable). So there exists no more than one acceptable partition.

We now show that there actually exists an acceptable partition. Since we have already defined above a recursive construction producing A, B, we just need to check that the resulting partition is in fact acceptable. Suppose not. Then there exist a smallest m such that $n+m=f_i$ for some $n<m$, and m, n are in the same set (by renaming A and B if necessary we may assume m, n are in A), i.e. the first m for which there is a problem with the above recursive construction. Then if f_j is as before the smallest Fibonacci number bigger than m we have $f_{j-1} \leq m < f_j$ which together with $n<m$ gives $f_{j-1} < m+n < 2f_j < f_{j+2}$. But $m+n=f_j$ is excluded by construction (recall that m is assigned to the set other than that of $f_j - m$). So it must be that $m+n=f_{j+1}$. On the other hand for $\hat{m}=f_j-m, \hat{n}=f_j-n$ we have:

1. $\hat{n} > \hat{m} > 0$ and $\hat{n} = f_j - n < m$, so the bigger of $\hat{n}, \hat{m}$ is less than the bigger of m, n.
2. $\hat{m}+\hat{n} = f_j + f_j - (m+n) = f_j + f_j - f_{j+1} = f_j - f_{j-1} = f_{j-2}$.
3. $\hat{m}$ is in B by construction.
4. $\hat{n}$ is in B, because otherwise $\hat{n}, n$ will be a pair of elements of A adding up to a Fibonacci number with the maximal element in the pair less than m, contradicting our choice of m.

These observations together mean that $\hat{n}, \hat{m}$ is a pair of elements of B adding up to a Fibonacci number with the maximal element in the pair less than m, contradicting our choice of m. This contradiction shows that A, B constructed above is indeed an acceptable partition.

4. The circle ω is drawn through the vertices A and B of the triangle ABC. If ω intersects AC at point M and BC at point P. The segment MP contains the center of the circle inscribed in ABC. Given that $AB=c$, $BC=a$ and $CA=b$, find MP.

Solution 1: Since $AMPB$ is cyclic, $\angle CMP = \angle CBA$ and $\angle CPM = \angle CAB$. Therefore the triangle CMP is similar to the triangle CBA. Let $CM = xCB = xa$, where x is the coefficient of similarity. Then $CP = xCA = xb$, $MP = xAB = xc$. If I is the center of the circle inscribed in ABC and r - its radius, we have the following equality for the areas: $S_{CMP} = S_{CMI} + S_{CPI}$. But $S_{CMP} = \frac{1}{2}xaxb\sin C$, and $S_{CPI} = \frac{1}{2}xbr$, $S_{CPM} = \frac{1}{2}xar$. Plugging these in and solving for x we get $x = frac(b+a)rba\sin C$. But $\frac{1}{2}ab\sin C = S_{ABC} = (a+b+c)r$. Hence $x = \frac{a+b}{a+b+c}$, $MP = \frac{c(a+b)}{a+b+c}$.

Solution 2: Keeping the notation from the first solution, we have

$$MP = MI + IP = r\left(\frac{1}{\sin B} + \frac{1}{\sin A}\right) = r\left(\frac{ac}{2S_{ABC}} + \frac{bc}{2S_{ABC}}\right) = \frac{c(a+b)}{a+b+c}.$$

5. For which n is it possible to fill the n by n table with 0's, 1's and 2's so that the sums of numbers in rows and columns take all different values from 1 to $2n$?

For odd n it is impossible to create a table like that. Indeed, in such a table the sum of all column sums and row sums would be $1+2+\ldots+2n = n(2n+1)$, and so would be odd. But it would also be twice the sum of all the numbers in the table (each number counted twice - once in the column sum, once in the row sum), and so would have to be even. This contradiction proves that no such table exists.

For any even n such a table exists. If one takes $2k$ by $2k$ table and fills in all the elements above the main diagonal (running from upper left to lower right corner of the table) with 2's, all the elements below the main diagonal with 0's, first k elements on the diagonal with 1s and the last k elements on the main diagonal with 2's one gets a table that satisfies the conditions. Indeed, the sums of elements in the first k rows (top to bottom) are $4k-1$, $4k-3$, ... , $2k+1$, the next k rows - $2k$, $2k-2$, ... ,2. The sums of elements in the first k columns (left to right) are 1, 3, ... , $2k-1$, the next k columns - $2k+2$, ... , $4k$. We see that all the numbers from 1 to $4k$ appear exactly once, as wanted.